Levi-Civita Connection

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On a Riemannian manifold, there is a natural choice for which connection to use. Choosing a connection is choosing a sense of acceleration on our manifold. For a Riemannian manifold M, a natural choice is to agree that geodesics have 0 accelertaion. Indeed, geodesics are paths that go in a "straight line" without changing velocity. Thus we would like a connection ∇ such that for any geodesic $\gamma(t)$ we have $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$. If we have $\gamma(t) = (x^1(t), \cdots, x^n(t))$ in local coordinates, this requirement is equivalent to

$$\begin{aligned}
0 &= \nabla_{\dot\gamma(t)}\dot\gamma(t) && (1)\\
&= \nabla_{\dot\gamma(t)}(\dot x^j(t)\partial_j) && (2)\\
&= \dot\gamma(t)(\dot x^j(t))\partial_j + \dot x^j(t)\nabla_{\dot\gamma(t)}\partial_j && (3)\\
&= \ddot x^j(t)\partial_j + \dot x^j(t)\nabla_{\dot x^i\partial_i}\partial_j && (4)\\
&= (\ddot x^k(t) + \dot x^i\dot x^j(t)A^k_{ij})\partial_k. && (5)
\end{aligned}$$

That is, this is equivalent to

$$\ddot x^k(t) + \dot x^i\dot x^j(t)A^k_{ij} = 0 \quad \text{for all } k.$$

Thus we want to choose our functions A^k_{ij} so that all geodesics satisfy the above equation. However, comparing the above equation to the geodesic equation, we realize that this will always be true if we choose $A^k_{ij} = \Gamma^k_{ij}$ to be the Christoffel symbols! Therefore, on a Riemannian manifold we should define our connection in coordinates by setting

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k. \qquad \text{(C)}$$

This definition, however, opens one important question: does this choice of connection rely on the choice of coordinates? The answer is no, which we will prove by later defining this natural connection in a coordinate independent way. The resulting connection is called the "Levi-Civita connection".

Recall the connection for Euclidean space $\mathbb{R}^n$ is given by

$$\nabla_X Y = XY^i\partial_i.$$

This *Euclidean connection* satisfies the product rule

$$\nabla_Z\langle X, Y\rangle = \nabla_X\left(\sum_i X^iY^i\right) = \sum_i (XZ^i)Y^i + X^i(ZY^i) = \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$$

Additionally, observe that for any vector fields X, Y over $\mathbb{R}^n$, we can compute the difference of the Euclidean connection $\nabla_X Y - \nabla_Y X$ to be

$$\nabla_X Y - \nabla_Y X = XY^i\partial_i - YX^i\partial_i = (XY^i - YX^i)\partial_i.$$

But this is juts the Lie bracket $[X, Y]$. That is, the Euclidean connection satisfies the commutator relation

$$\nabla_X Y - \nabla_Y X = [X, Y]. \qquad \text{(S)}$$

It turns out that for a Riemannian manifold M, the Levi-Civita connection defined in local coordinates by (C) also satisfies the commutator relation (S) as well as the following product rule with respect to the metric g.

$$\nabla_Z \langle X, Y\rangle_g = \langle \nabla_Z X, Y\rangle_g + \langle X, \nabla_Z Y\rangle_g. \tag{M}$$

A connection satisfying (S) is called *symmetric* and a connection satisfying the product rule (M) is said to be *compatible with the metric*. We begin by showing the symmetry which is simply a coordinate computation.

Prop. The Levi-Civita connection as defined in coordinates by (C) is symmetric.

Proof. Compute

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= \nabla_{X^i\partial_i}(Y^j\partial_j) - \nabla_{Y^j\partial_j}(X^i\partial_i) && (6)\\
&= X^i((\partial_i Y^j)\partial_j + Y^j\nabla_{\partial_j}\partial_i) - Y^j((\partial_j X^i)\partial_i + X^i\nabla_{\partial_i}\partial_j) && (7)\\
&= (XY^j\partial_j - YX^i\partial_i) + X^iY^j(\nabla_{\partial_j}\partial_i - \nabla_{\partial_i}\partial_j)\partial_k && (8)\\
&= [X,Y] + X^iY^j(\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k. && (9)
\end{aligned}$$

Then the result follows from $\Gamma^k_{ij} = \Gamma^k_{ji}$ which we can see from the definition of Christoffel symbols:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$$

□

Next we show the Levi-Civita connection as defined in coordinates is compatible with the metric, which follows from a substantially longer coordinate computation.

Prop. The Levi-Civita connection as defined in coordinates by (C) is compatible with the metric.

Proof. First we expand out the right side of (M).

$$\begin{aligned}
\langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle &= \langle \nabla_{Z^k\partial_k}(X^i\partial_i), Y^j\partial_j\rangle + \langle X^i\partial_i, \nabla_{Z^k\partial_k}(Y^j\partial_j)\rangle && (10)\\
&= Z^k(Y^j\langle \partial_k X^i\partial_i + X^i\nabla_{\partial_k}\partial_i, \partial_j\rangle + X^i\langle \partial_i, \partial_k Y^j\partial_j + Y^j\nabla_{\partial_k}\partial_j\rangle) && (11)\\
&= Z^k(Y^j\langle(\partial_k X^l + X^i\Gamma^l_{ik})\partial_l, \partial_j\rangle + X^i\langle \partial_i, (\partial_k Y^l + Y^j\Gamma^l_{kj})\partial_l\rangle) && (12)\\
&= Z^kY^j(\partial_k X^l + X^i\Gamma^l_{ik})g_{lj} + Z^kX^i(\partial_k Y^l + Y^j\Gamma^l_{kj})g_{il} && (13)\\
&= Z^k(Y^j\partial_k X^l g_{lj} + X^i\partial_k Y^l g_{il}) + Z^kX^iY^j(\Gamma^l_{ik}g_{lj} + \Gamma^l_{kj}g_{il}). && (14)
\end{aligned}$$

Next we expand out the left size of (M).

$$\nabla_Z\langle X, Y\rangle = Z^k\partial_k\langle X^i\partial_i, Y^j\partial_j\rangle = Z^k\partial_k(X^iY^jg_{ij}) = Z^k(Y^j\partial_k X^i g_{ij} + X^i\partial_k Y^j g_{ij}) + Z^kX^iY^j\partial_k g_{ij}.$$

Note these expansions are quite similar, and we see that in fact the right and left sides of (M) are equal so long as we can show

$$\partial_k g_{ij} = \Gamma^l_{ik}g_{lj} + \Gamma^l_{kj}g_{il}.$$

Indeed, to show this we use the definition of the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$$

and apply the matrix g_{km} to both sides to conclude

$$\Gamma^k_{ij}g_{km} = \frac{1}{2}(\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij}).$$

Thus using the above expression twice we can compute

$$\Gamma^l_{ik}g_{lj} + \Gamma^l_{kj}g_{il} = \frac{1}{2}(\partial_i g_{kj} + \partial_k g_{ji} - \partial_j g_{ik}) + \frac{1}{2}(\partial_k g_{ji} + \partial_j g_{ik} - \partial_i g_{kj}) = \partial_k g_{ij}$$

as needed.

□

It turns out that these two properties – symmetry and metric compatibility – are quite special. In fact, on a Riemannian manifold there will only be one connection that satisfies both properties.

Prop. (Fundamental Theorem of Riemannian Geometry). For any Riemannian manifold M, there exists a unique connection ∇ that is both symmetric and compatible with the metric. This connection is called the *Levi-Civita connection*.

Proof in coordinates. We have already demonstrated existence, for the Levi-Civita connection is symmetric and metric-compatible. To see why an arbitrary symmetric and metric-compatible connection ∇ must be the Levi-Civita connection, we work locally in coordinates (x^i) and write $\nabla_{\partial_i}\partial_j = A^k_{ij}$. By the same computation we performed to show symmetry of the Levi-Civita connection, we see the symmetry of ∇ is equivalent to $A^k_{ij} = A^k_{ji}$. Similarly, we see ∇ is compatible with the metric exactly when

$$\partial_k g_{ij} = A^l_{ik} g_{lj} + A^l_{kj} g_{il}.$$

by the corresponding computation for the Levi-Civita connection; this expression is often called the *first Christoffel identity*. These two requirements give a linear system of $\frac{1}{2}n^2(n+1)$ equations with the same amount of unknowns. The trick to solve this system is to permute the first Christoffel identity to get cancellation and solve for the sum

$$\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} = (A^p_{ij} g_{pl} + A^p_{il} g_{jp}) + (A^p_{ji} g_{pl} + A^p_{jl} g_{ip}) - (A^p_{li} g_{pj} + A^p_{lj} g_{ip}) = 2A^p_{ij} g_{pl}. \tag{15}$$

Then applying the inverse matrix g^{kl} we recover the definition of the Christoffel symbols:

$$A^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

□

Proof without coordinates. Existence follows from the Levi-Civita connection. For uniqueness, suppose ∇ is a symmetric and metric-compatible connection and use both properties to write

$$X\langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_Z X\rangle_g + \langle Y, [X, Z]\rangle_g. \tag{16}$$

We will use a similar trick as the proof in coordinates to find an expression for ∇. By cyclically permuting the above, we get two more identities:

$$Y\langle Z, X\rangle_g = \langle \nabla_Y Z, X\rangle_g + \langle Z, \nabla_Y X\rangle_g = \langle \nabla_Y Z, X\rangle_g + \langle Z, \nabla_X Y\rangle_g + \langle Z, [Y, X]\rangle_g \tag{17}$$

$$Z\langle X, Y\rangle_g = \langle \nabla_Z X, Y\rangle_g + \langle X, \nabla_Z Y\rangle_g = \langle \nabla_Z X, Y\rangle_g + \langle X, \nabla_Y Z\rangle_g + \langle X, [Z, Y]\rangle_g. \tag{18}$$

Now adding the first two equations and subtracting the third gives the cancellation

$$X\langle Y, Z\rangle_g + Y\langle Z, X\rangle_g - Z\langle X, Y\rangle_g = 2\langle \nabla_X Y, Z\rangle_g + \langle Y, [X, Z]\rangle_g + \langle Z, [Y, X]\rangle_g - \langle X, [Z, Y]\rangle_g. \tag{19}$$

Thus we can solve for $\langle \nabla_X Y, Z\rangle_g$ to find

$$\langle \nabla_X Y, Z\rangle_g = \frac{1}{2}(X\langle Y, Z\rangle_g + Y\langle Z, X\rangle_g - Z\langle X, Y\rangle_g - \langle Y, [X, Z]\rangle_g - \langle Z, [Y, X]\rangle_g + \langle X, [Z, Y]\rangle_g). \tag{20}$$

which uniquely determines the connection ∇. The above is thus a coordinate-invariant expression for the Levi-Civita connection and is called *Koszul's formula*.