Covariant Derivative

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Differentiating vector fields in $\mathbb{R}^n$

Given a path $\gamma(t)$ in $\mathbb{R}^n$, we know we can represent its velocity by

$$\dot{\gamma}(t) = \dot{\gamma}^i(t)\partial_i.$$

Furthermore, in the case of $\mathbb{R}^n$, we understand the acceleration of the curve $\gamma(t)$ to be given by

$$\ddot{\gamma}(t) = \frac{d}{dt}\left(\dot{\gamma}^i(t)\partial_i\right) = \ddot{\gamma}^i(t)\partial_i.$$

In computing the acceleration, we are differentiating a vector field. More generally in $\mathbb{R}^n$, we understand the derivative of some vector field X in the direction $v \in T_p\mathbb{R}^n$ at point p, which we denote $\nabla_v X$, to be

$$\nabla_v X = \frac{d}{dt}\bigg|_{t=0} X(\gamma(t)) = \frac{d}{dt}\bigg|_{t=0} X^i(\gamma(t))\partial_i = v(X^i)\partial_i.$$

where $\gamma(t)$ is a smooth curve such that $\dot{\gamma}(0) = v$. This directional derivative enjoys the following nice properties:

1. First, the directional derivative is linear with respect to the direction. Indeed, for any vectors v, w based at point $p \in \mathbb{R}^n$, real numbers a, b, and vector field X we find

$$\nabla_{av+bw} X = (av + bw)X^i\partial_i = avX^i\partial_i + bwX^i\partial_i = a\nabla_v X + b\nabla_w X.$$

2. Second, this directional derivative is linear with respect to the vector fields. Indeed, for any vector v based at $p \in \mathbb{R}^n$, any real numbers a, b, and any vector fields X, Y that

$$\nabla_v(aX + bY) = v(aX^i + bY^i)\partial_i = avX^i\partial_i + bvY^i\partial_i = a\nabla_v X + b\nabla_v Y$$

3. Finally, we have a product rule. For any vector v at point $p \in \mathbb{R}^n$, vector field X, and smooth function f we find

$$\nabla_v(fX) = v(fX^i)\partial_i = (vf)X^i\partial_i + f(vX^i)\partial_i = (vf)X + f\nabla_v X$$

The problem with differentiating vector fields on manifolds

However, the definition of the directional derivative of a vector field X in direction v

$$\frac{d}{dt}\bigg|_{t=0} X(\gamma(t)) = \lim_{t\to 0} \frac{X(\gamma(t)) - X(\gamma(0))}{t}$$

does not make sense on a general Riemannian manifold! The vectors $X(\gamma(t))$ and $X(\gamma(0))$ belong to the two different tangent spaces $T_{\gamma(t)}M$ and $T_{\gamma(0)}M$, so it does not make sense to subtract them. Note that in the case of $\mathbb{R}^n$, there is a natural identification between different tangent spaces $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^n$ by simply translating the vectors. However, this a consequence of having a nice coordinate frame (∂_i), for we are naturally identifying $v^i\partial_i \in T_p\mathbb{R}^n$ with $v^i\partial_i \in T_q\mathbb{R}^n$.

Connections

We are looking to define a directional derivative operation $\nabla_v X$ on a general smooth manifold. Formally, we are looking for a map $\nabla : T_pM \times \Gamma(TM) \to T_pM$ varying smoothly with p. If this map is denoted $\nabla : (v, X) \mapsto \nabla_v X$, we want ∇ to satisfy the following three properties we expect from a directional derivative. In the following, $a, b \in \mathbb{R}$ are real numbers, $v, w \in T_pM$ are vectors based at p, and $X, Y \in \Gamma(TM)$ are vector fields on the manifold.

1. Linearity with respect to the direction:
$$\nabla_{av+bw} X = a\nabla_v X + b\nabla_w X.$$
2. Linearity with respect to the vector field:
$$\nabla_v(aX + bY) = a\nabla_v X + b\nabla_v Y.$$
3. Product rule:
$$\nabla_v(fX) = (vf)X + f\nabla_v X.$$

Such a map $\nabla : T_pM \times \Gamma(TM) \to T_pM$ varying smoothly with p that satisfies the three properties above is called a *connection*. Equivalently, we can consider two vector fields X, Y and say $\nabla_X Y$ is vector field such that $(\nabla_X Y)(p) = \nabla_{X(p)} Y$ so that we now have a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$. The directional derivative $(\nabla_X Y)$ corresponding to a connection is called the *covariant derivative* of Y in the direction of X. The main question, however, is how do we choose which connection to use? In general, there are many connections on a smooth manifold: choose any coordinate frame (∂_i) and decide the derivative of each coordinate frame in the direction of all the other coordinate frames at every point. That is, choose functions A_{ij}^k such that
$$\nabla_{\partial_i}\partial_j = A_{ij}^k \partial_k.$$
Then given any arbitrary vector fields $X = X^i\partial_i$ and $Y = Y^j\partial_j$, we can determine what the covariant derivative $\nabla_X Y$ should be by
$$\nabla_X Y = \nabla_{X^i\partial_i}(Y^j\partial_j) = X^i((\partial_i Y^j)\partial_j + Y^j\nabla_{\partial_i}\partial_j) = X^i(\partial_i Y^k + Y^j A_{ij}^k)\partial_k. \tag{A}$$

For any smooth functions A_{ij}^k that we choose, the above formula in fact defines a connection.

Exercise. Verify that for any smooth functions A_{ij}^k, the formula (A) defines a connection by checking the three necessary properties are satisfied.

Thus there are many possible connections on a general smooth manifold. For some intuition for connections, note that choosing a connection gives us a sense of acceleration on our manifold. Given a curve $\gamma(t)$, it's velocity at each point along the curve is given by the tangent vectors $\dot\gamma(t)$. Then the acceleration should be the change of $\dot\gamma(t)$ in direction $\dot\gamma(t)$. That is, the *acceleration* of $\gamma(t)$ is defined to be $\ddot\gamma(t) = \nabla_{\dot\gamma(t)}\dot\gamma(t)$.