Topological K-Theory for Undergraduates

Sean Richardson

Mathematics and Physics '20

Advised by Dr. Iva Stavrov

Lewis & Clark College, Department of Mathematical Sciences

Contents

Introduction

This expository document aims to make the basics of topological K-theory accessible to undergraduate students. The document assumes the reader has had first classes in linear algebra and discrete mathematics, knows the basics of group theory and ring theory, and is comfortable reading formal proofs. The material on K-theory is centered around Allen Hatcher's Book "K-theory and Vector Bundles" [**4**].

K-theory considers objects called vector bundles, which are created by placing a vector space at every point of a topological object while adhering to some technical rules. For instance, place the vector space $\mathbb{R}$ on every point on the circle so that each copy of $\mathbb{R}$ points in the same direction — this forms a cylinder. Now place the copies of $\mathbb{R}$ onto the circle while slowly changing the orientation such that after rounding the circle, the vector spaces have made a half twist — this forms a Mobius Band. K-theory fixes a topological space and considers all possible vector bundles over that topological space. This collection of vector bundles over the circle would include the cylinder and the Mobius band. The objective of K-theory is to craft this collection of all possible vector bundles into a ring so that two vector bundles can be added or multiplied together. This process would associates a ring with the original fixed topological space. However, there is some work to do before getting to this point.

Chapter 1 introduces the basics of category theory, explaining that rings, topological spaces, and vector spaces are all categories, and pointing out that a more precise formulation of "associating

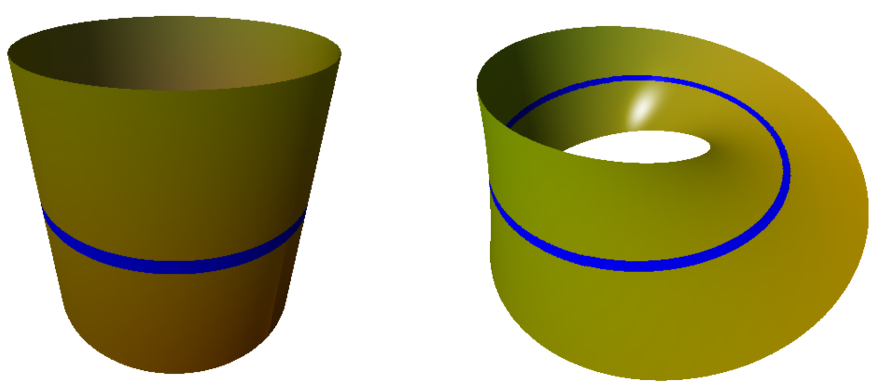

Figure 1. Cylinder and Mobius Band are Vector Bundles over the Circle

a ring with a topological space" is through the use of functors — a mapping from one category to another. All later chapters will practice the language of category theory.

Chapter 2 introduces the algebra necessary for later chapters, beginning by anticipating a future issue; the formulation of K-theory will get stuck stuck with a ring without additive inverses such as the nonnegative integers $\mathbb{N} \cup \{0\}$. However, just as the nonnegative integers naturally rest inside the complete ring of integers, these broken rings with particular properties can be extended into complete rings. This chapter also introduces the direct sum and tensor product through their universal properties, which are the key to defining the addition and multiplication operations on vector bundles.

Chapter 3 is a primarily self contained chapter on all of the topology that will be necessary later on. This includes basic definitions, relevant examples, and some particular operations on topological objects.

Chapter 4 gives a formal introduction to vector bundles, working towards the definition of addition and multiplication between vector bundles by extending the direct sum and tensor product operations from vector spaces to vector bundles. This chapter additionally introduces pullback bundles and some miscellaneous results on vector bundles required later.

In Chapter 5, it becomes clear that only compact Hausdorff spaces will guarantee the pleasant ring properties and individual vector bundles cannot form a ring; rather, the elements of the ring must be equivalence classes. The choice of equivalence class gives rise to either K-theory and reduced K-theory, which are both formally defined in this chapter. This completes the goal of associating a ring to a topological object.

Chapter 6 searches for extra structure on these induced K-theory rings. K-theory has particularly nice properties over spheres such as the Bott periodicity theorem. This allows for a space-subspace pair to generate an infinite sequence of groups that satisfies all the properties of a cohomology theory. An external product is then defined on K-theory and reduced K-theory, which gives useful information about the induced K-theory rings.

Chapter 7 focuses on an application of K-theory — that there are only finite real division algebras. The existence of a division algebra in $\mathbb{R}^n$ can be reduced to a problem about spheres, which K-theory is well equipped to study. The information given by the external product is enough to rule out the existence of odd dimensional division algebras and the idea of the even dimensional case is discussed.

CHAPTER 1

Category Theory

1. Categories and Functors

Begin by considering the following familiar mathematical subjects that do appear to have any connection with one another. The study of linear algebra focuses on two things: vector spaces and linear transformations between vector spaces. Set theory examines sets as well as mappings between sets. Group theory considers groups and homomorphisms between groups whereas ring theory focuses on rings and homorphisms between rings. Real analysis studies metric spaces together with continuous functions between metric spaces as well as manifolds paired with smooth mappings between these manifolds. A pattern emerges; each one of these topics have two things — some *objects* of study (vector spaces, metric spaces, manifolds, sets, groups, rings) together with some type of *morphism* between the objects of study (linear transformations, continuous functions, smooth mappings, set mappings, homomorphisms). Any such object-morphism pair is called a *category* so long as it obeys some rules.

Definition 1.1 (Category)**.** Let $\mathcal{O}$ denote a collection of objects and let $\mathcal{M}$ denote a collection of morphisms. Then, the pair $(\mathcal{O}, \mathcal{M})$ is called a *category* if:

(i) There is an identity element Id in the morphisms $\mathcal{M}$ that satisfies: $\text{Id}(obj) = obj$ for all objects obj in the objects $\mathcal{O}$ and the composition law $f \circ \text{Id} = f = \text{Id} \circ f$ holds for all f in $\mathcal{M}$.
(ii) For a specific object obj, the collection of morphisms from obj to itself must contain the identity
(iii) composition is associative: for all $f, g, h \in \mathcal{M}$, $(f \circ g) \circ h = f \circ (g \circ h)$.

The formal definition of a category aims to take all the specific object-morphisms pairs mentioned earlier and identify the key commonalities between them. To get a feel for this formal notion of category, examine the following two categories.

Example 1.2 (Vector Spaces as a Category)**.** The category of vector spaces takes the collection of all vector spaces as objects and all linear maps between vector spaces as morphisms.

(i) The collection of all linear maps indeed includes the identity mapping. Here, the 1×1 identity matrix, the 2×2 identity matrix, the 3×3 identity matrix, and all of the others are representations of the same identity morphism. Indeed for any other linear map L, the composition requirement $\text{Id} \circ L = L = L \circ \text{Id}$ holds.
(ii) The collection of all linear maps from a particular vector space V to itself indeed includes the identity. In this case, fixing a basis for an n dimensional vector space allows the $n \times n$ identity matrix to represent this identity map.
(iii) The composition of linear maps, by the nature of functions, is associative.

Example 1.3 (Rings as a Category)**.** The category of rings takes the collection of all rings as objects and the collection of all rings homomorphisms between rings as morphisms.

(i) The collection of all ring homomorphisms indeed includes the identity, which indeed satisfies the composition requirement $\varphi \circ \mathrm{Id} = \varphi$.
(ii) Additionally, the set of all ring homomorphisms from a particular ring R to itself includes the identity mapping. Taking the elements of R to themselves is a valid ring homomorphism from R to R.
(iii) Finally, the composition of ring homomorphisms, by the nature of functions, is associative.

So the rules posed in the definition of category seem to work out for specific examples. Pinning down the similarities between different categories allows for creating relationships between categories. A *functor* is a way to map one category into another.

Definition 1.4 (Functor)**.** Consider two categories $\mathcal{C}_A = (\mathcal{O}_A, \mathcal{M}_A)$ and $\mathcal{C}_B = (\mathcal{O}_B, \mathcal{M}_B)$. Next consider the mapping $\mathcal{F} : \mathcal{C}_A \to \mathcal{C}_B$, which maps $\mathcal{O}_A$ to $\mathcal{O}_B$ and $\mathcal{M}_A$ to $\mathcal{M}_B$. Then, $\mathcal{F}$ is called a *functor* if $\mathcal{F}$ preserves identity identity: $\mathcal{F}(\mathrm{Id}_A) = \mathrm{Id}_B$ as well as satisfies either one of the following two composition requirements:

- For, $f, g \in \mathcal{M}_{\mathcal{A}}$, then $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$. Here, $\mathcal{F}$ is called a *covariant functor*.
- For, $f, g \in \mathcal{M}_{\mathcal{A}}$, then $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$. Here, $\mathcal{F}$ is called a *contravariant functor*.

A simple example of a functor is included below.

Example 1.5. Groups also form a category by taking morphisms to be the usual homomorphisms of groups: a function that preserves the operation. Then, a functor $\mathcal{F}$ from the category of rings to the category of groups can be defined as follows. For any ring $(R, +, \cdot)$, let $\mathcal{F}((R, +, \cdot))$ be the group $(R, +)$. Additionally, let the functor $\mathcal{F}$ take a ring homomorphism φ to the same mapping φ, but now viewed as a group homomorphism. With this definition, the identity will surely be mapped to the identity under the functor, and the covariant composition rule will apply.

This functor is quite boring, for it simply "forgets" the ring structure. But do not worry — the next five chapters are dedicated to constructing a much more interesting functor. K-theory is a contravariant functor from the category of topological spaces to the category of rings.

The difference between covariant and contravariant functors becomes more clear when examining *commutative diagrams* as depicted in the included figures. Figure 1 shows the arrows pointing in the same direction and corresponds to a covariant functor. Figure 2, however, reverses the direction of the arrow with the application of the functor and represents a contravariant functor. Both of these diagrams represents a functor between two categories, say from category A to category B. In the diagrams, X and Y represent two objects in category A and f represents a morphism from object X to object Y. Then $\mathcal{F}$ denotes a functor from category A to category B and so $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are objects in categories B; more specifically, $\mathcal{F}(X)$ is where the functor maps object X to and $\mathcal{F}(Y)$ is where the functor maps object Y to. The functor takes f to $\mathcal{F}(f)$, which represents a morphism between $\mathcal{F}(X)$ and $\mathcal{F}(Y)$, but keep in mind the direction of this mapping depends on the type of functor.

So, the direction of the arrows is preserved for covariant functors and reversed for contravariant functors. But how does this but how does this relate to the composition requirements as given in definition 1.4? Applying the functor on an additional object Z together with an additional morphism f as in Figures 3 and 4 gives a visual of the composition requirements. For covariant

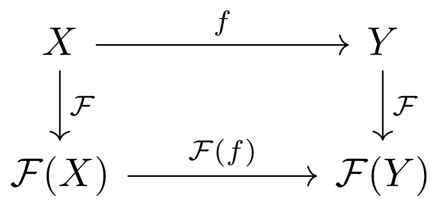

FIGURE 1. Covariant Functor

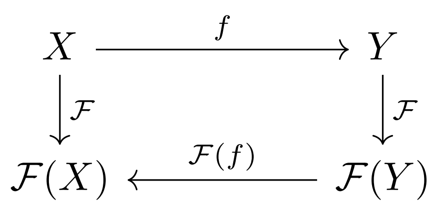

FIGURE 2. Contravariant Functor

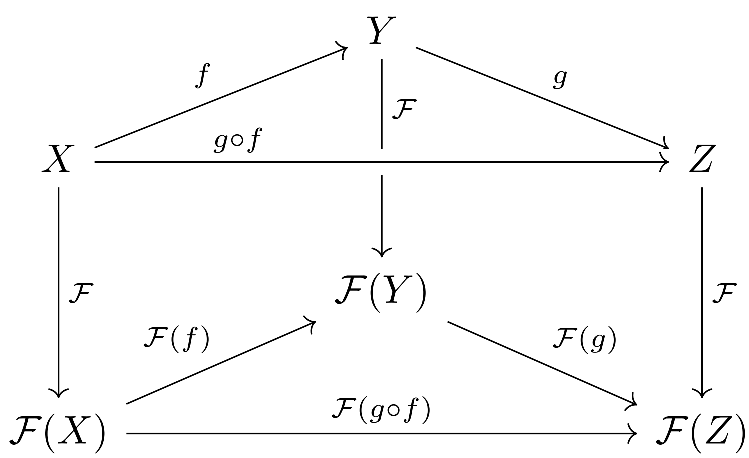

FIGURE 3. Covariant Functor Composition

functors as in figure 3, the natural composition requirement is not surprising, $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$. However, in the case of contravariant functors as depicted in Figure 4, the statement $\mathcal{F}(g) \circ \mathcal{F}(f)$ does not make sense. It is impossible to apply $\mathcal{F}(f)$ and then immediately $\mathcal{F}(g)$ because the input space of $\mathcal{F}(g)$ is different than the output space of $\mathcal{F}(f)$. The relevant composition requirement then must be $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

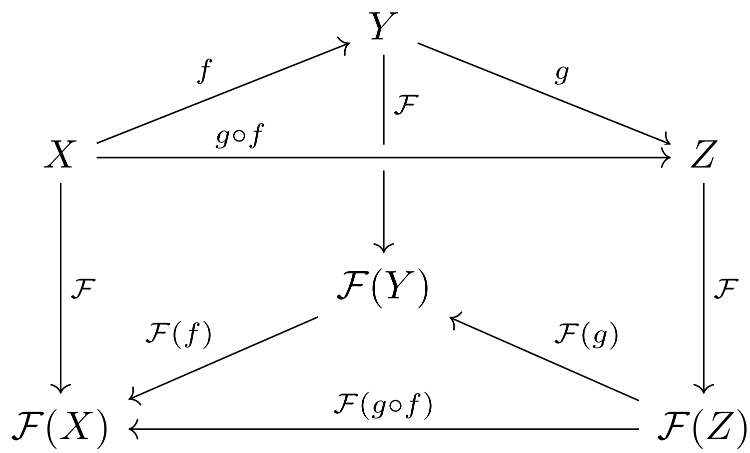

FIGURE 4. Contravariant Functor Composition

The word isomorphism is used when working in rings, groups, manifolds, vector spaces, and various other settings. So perhaps it does not come as a surprise that category theory also provides general definition of isomorphism that carries over to all of these different categories.

Definition 1.6 (Isomorphism)**.** Given a category $(\mathcal{O}, \mathcal{M})$ and two objects X and Y, then a morphism $\varphi : X \to Y$ is an *isomorphism* if there exists a morphism $\psi : Y \to X$ such that $\varphi \circ \psi = \text{Id}$ and $\psi \circ \varphi = \text{Id}$.

The above definition states an isomorphism is a morphism that has a morphism as an inverse. For example, a linear map is an isomorphism if it has a linear inverse and a ring homomorphism is an isomorphism if its inverse is a ring homomorphism. However, in linear algebra, the definition of an isomorphism is often given as a bijective linear map. Similarly, in ring theory, an isomorphism is often defined as a bijective ring homomorphism. These definitions do not address the morphism properties of the inverse. However, in these specific cases, one can verify that the inverse of a bijective linear map L is always linear. For instance, the scalar verification would go as follows.

$$L^{-1}(\alpha x) = L^{-1}(\alpha L(x')) = L^{-1}(L(\alpha x')) = \alpha x' = \alpha L^{-1}(x)$$

Where the first step $x = L(x')$ for some x' uses surjectivity of L and the last step $x' = L^{-1}(x)$ uses the injectivity of L. A similar argument gives the additive property of linear transformations and the ring homomorphism properties. This demonstrates that the familiar definitions of isomorphism are equivalent to this abstract version. However, it is not always true that a bijective morphism will have a morphism as an inverse. In particular, a continuous bijection need not have a continuous inverse.

2. Further Examples

This section will examples of two more categories along with a functor between these two categories. The category of real division algebras considers the following objects.

Definition 1.7 (Division Algebra)**.** A real division algebra considers a set $\mathbb{R}^n$ for some n together with operations $(+, \cdot)$ such that $(\mathbb{R}^n, +)$ forms an abelian group, there is a multiplicative identity, and every element has a two-sided multiplicative inverse.

Note that division algebras do not require commutativity or associativity the multiplication operation, which is what distinguishes the definition of division algebras from that of fields. Every field is then a division algebra, including the fields $\mathbb{R}$ and $\mathbb{C}$. However, there are examples of division algebras that are not fields.

Such an example of a non-commutative division algebra the *quaternions*, which considers the set $\mathbb{R}^4$ and denotes each element $(a, b, c, d) \in \mathbb{R}^4$ by $a + bi + cj + dk$. Applying the relations

$$i^2 = j^2 = k^2 = ijk = -1$$

makes the set $\{\pm 1, \pm i, \pm j, \pm k\}$ a (noncommutative) group under multiplication. With this, multiplication of two elements of $\mathbb{R}^4$ is given by expanding the expression

$$(a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k)$$

with the distribution law, and simplifying using the operations of the group $\{\pm 1, \pm i, \pm j, \pm k\}$. With this construction, every element indeed has a multiplicative inverse. Specifically,

$$(a + bi + cj + dk)^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

Thus $\mathbb{R}^4$ can be equipped with this division algebra structure. The *Cayley octonions* are an example of a non-associative division algebra structure on the set $\mathbb{R}^8$. Considering the division algebras over the objects

Example 1.8 (Division Algebras as a Category)**.** Real division algebras indeed form a category. A morphism between two division algebra objects is defined to be any function that preserves the additive operation, the multiplicative operation, and the identity element. And so, as in the case of rings, the identity operation is indeed a morphism that is included in all morphisms from a real division algebra to itself. Associativity follows from the nature of functions.

The category of H-spaces on spheres will consider the following objects.

Definition 1.9 (Sphere as an H-Space)**.** Let the sphere S^{n-1} denote the subset of $\mathbb{R}^n$ that is distance 1 from the origin. Then, S^{n-1} is an *H-space* if it is equipped with a continuous multiplication map $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ that has a two sided identity element e.

An H-space is given by taking S^1 as the subset of complex numbers with norm 1. This equips S^1 with a continuous multiplication map given by multiplication of complex numbers. This map is closed in S^1 because the product of two complex numbers with norm 1 will have norm 1 and thus this gives an H-space.

Example 1.10 (H-Spaces on Spheres as a Category)**.** A morphism between H-spaces is a continuous map that preserves the H-space multiplication operation and the identity element. Again, the identity map satisfies these conditions and is a valid morphism from an H-space to itself. Associativity follows from the nature of functions.

The complex numbers were used earlier to assign an H-space structure to S^1. This is representative of a larger functor from the category of real division algebras to the category of H-spaces over spheres.

Example 1.11. Take a real division algebra over $\mathbb{R}^n$. Next take the sphere S^{n-1} and consider the sphere as the subset of $\mathbb{R}^n$ with all points a distance of 1 from the origin. The product of two elements of a division algebra with distance 1 from the origin will return an element of distance 1 from the origin, and thus this induces a closed map $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ that satisfies the necessary properties of an H-space multiplication by the given properties of the division algebra multiplication. Further, the identity element in the division algebra must be of distance 1 from the origin and thus will be contained in the H-space. Because the H-space multiplication is only a restriction of the the division algebra multiplication to a smaller area, the identity homomorphism of division algebras induces the identity homomorphism for H-spaces and morphisms obey the covariant commutativity law.

Category theory is, by design, abstract. Comfort with speaking in the language of category theory comes with practice and the following chapters will aid in practicing this language.

CHAPTER 2

Algebra

1. Ring Completion

An example of a category that the reader is likely unfamiliar with is the category of semirings. The objects in these categories are called semirings, which are simply rings without necessarily having an additive inverse.

Definition 2.1 (Semiring)**.** A *semiring* is a set S paired with the binary operations $(+, \cdot)$ such that the following properties hold:

(i) The operation $+$ is associative and commutative
(ii) The operation $\cdot$ is associative
(iii) The operation $\cdot$ distributes over $+$
(iv) S has both an additive and multiplicative identity.

A simple example of a semiring is the set of nonnegative integers under the usual addition and multiplication operations. The element 0 is the additive identity and 1 is the multiplicative identity. In fact, this example of $\mathbb{N} \cup \{0\}$ has two additional nice properties: commutativity of multiplication and the cancellation property under addition. To be precise, the cancellation property promises that given elements a, b, and s in a semiring, the statement $a+s = b+s$ implies $a = b$. This section will focus on commutative semirings with the additive cancellation property.

To complete the category of semirings, the morphisms of a category must be discussed. In this case, the morphisms are referred to as homomorphisms of semirings which are defined as follows.

Definition 2.2 (Homomorphism of Semirings)**.** Take monoids S and R and consider a mapping $\varphi : S \to R$. Then, φ is a *homomorphism of semirings* if:

(i) $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in S$.
(ii) $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$
(iii) $\varphi(1) = 1$

Note homomorphisms between semirings follows the same structure as homomorphism between rings; in fact, a homomorphism of rings *is* a homomorphism of semiring, for rings are themselves semirings. In fact, even a mapping from a semiring S to a ring R could be considered a homomorphism of semirings if the mapping satisfies the necessary properties. Overall, the category of semirings is frustratingly close to the category of rings. Luckily, there is a functor from the category of commutative semirings with cancellation to the category of rings called *ring extension* — a way to expand the structure of a monoid into a fully fledged ring. K-theory heavily relies on this functor, so pay particular attention to it.

The formal definition of ring extension is addressed shortly, but first consider the following example. The semiring of nonnegative integers predictably extends into the ring of all integers. The idea of ring extension is that the semiring $\mathbb{N} \cup \{0\}$ naturally belongs inside the larger ring $\mathbb{Z}$. This idea of "belonging inside" a particular ring is formally given by *universal property* in the following definition.

Definition 2.3 (Ring Completion)**.** Take commutative semiring S with additive cancellation. Then, a *ring completion* of of S is a commutative ring R together with an injective homomorphism $i : S \to R$ that satisfies the following property: for any commutative ring R' and corresponding homomorphism of semirings $\varphi : S \to R'$, there exists a unique homomorphism of rings $\psi : R \to R'$ such that the following diagram commutes:

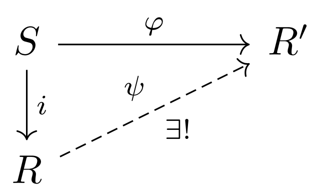

FIGURE 1. The Universal Property

That is, $\psi \circ i = \varphi$.

There is still work to be done with this definition; it must still be verified that the above construction exists and is unique. The requirement that the above triangle commutes is the *universal property*, and throughout this chapter there will be many constructions using the universal property structure.

To get a better feel for this definition, observe that the extension of the nonnegative integers into the integers fulfills this universal property. In this case, the extension function $i : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ is given by the injective identity function $i(n) = n$. Now let R' be an arbitrary ring and let $\varphi : \mathbb{N} \cup \{0\} \to R'$ be an arbitrary semiring homomorphism. Now note that for $\psi(i(1)) = \varphi(1)$ by definition, so $\psi(1) = \varphi(1)$. Because $\{1\}$ generates the integers, this defines a unique homomorphism φ. In particular, for any difference of nonnegative integers $m - n$, the homomorphism ψ can be written $\psi(n-m) = \varphi(n) - \varphi(m)$, which satisfies the universal property and thus gives the existence of the homomorphism. In fact, knowing that $\mathbb{Z}$ fulfills the universal property is enough to give that $\mathbb{Z}$ is *the* ring completion of the nonnegative integers, for the nature of the universal property condition forces the ring extension to be unique as demonstrated in the following proof.

PROOF OF UNIQUENESS OF DEFINITION 2.3. Consider two ring completions (R, i) and (R', i') of a semiring S. It must be shown that R and R' are isomorphic. By (R, i) a ring completion and taking (R', i') to be a ring-homomorphism pair, the universal property in the definition of ring completion promises the existence of a unique homomorphism $\psi_1 : R \to R'$ such that $\psi_1 \circ i = i'$. Similarly, by swapping the roles of (R, i) and (R', i'), there exists a a unique homomorphism $\psi_2 : R' \to R$ such that $\psi_2 \circ i' = i$. But then, the composition $\psi_2 \circ \psi_1 : R \to R$ satisfies $(\psi_2 \circ \psi_1) \circ i = i$. Thus $\psi_2 \circ \psi_1$ must be the unique map promised by the universal property by applying the universal property of ring completion (R, i) on (R, i) itself. However, the identity mapping also satisfies the condition $\mathrm{Id} \circ i = i$ and so the uniqueness conditions gives that $\psi_2 \circ \psi_1 = \mathrm{Id}$. See Figure 2 for a visual of this argument. The same argument gives that $\psi_1 \circ \psi_2 = \mathrm{Id}$ and thus ψ_1 and ψ_2 are inverses of one another. This gives that ψ_1 and ψ_2 are isomorphisms and so $R \cong R'$.

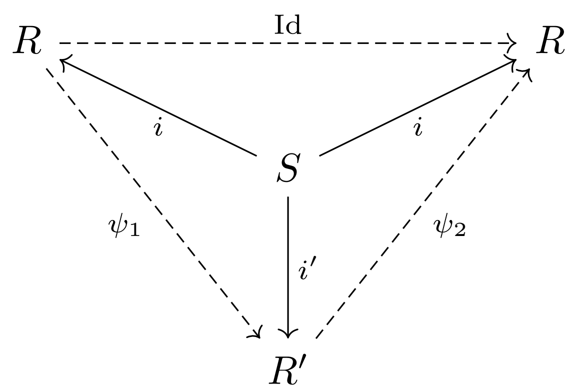

Figure 2. Uniqueness of Ring Completion Argument

□

The above argument never appeals to the specific properties of rings and semirings; in fact, this argument applies to *all* definitions defined through the universal property. For every additional formulation using the universal property, uniqueness will follow automatically.

All that needs to be shown to justify a definition using the universal property is existence. The existence proof is given at the end of this chapter, but note the following important lessons from the proof. The existence proof is related to the fact that each element in $\mathbb{Z}$ can be represented by a difference of nonnegative integers $a - b$. In a semiring, there is no promise of subtraction, but a pair (a, b) can secretly represent the difference $a - b \in \mathbb{Z}$ through an equivalence relation. For an arbitrary semiring S, the proof uses the equivalence relation $\sim$ on $S \times S$ given by $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. Again, think of this equivalence relation as "sneaky subtraction", stemming from the wish to express $a_1 - b_1 = a_2 - b_2$ without the explicit use of subtraction. There is a natural addition on the equivalence classes that gives a commutative group structure. However, in order to get a well-defined multiplication, the semiring must have the additive cancellation property.

Rings are nicer than semirings; they have additive inverses and extensive theory. As shown above, every commutative semiring with cancellation extends to a unique ring; therefore, given a semiring with these properties, it is best to ditch the semiring and instead talk about the ring extension. This idea of extending an "incomplete" object into a nicer object is the whole point of constructions using the universal property. The next section contains two more instances of extending an incomplete object into a nicer object.

2. Packing Together Modules

This section introduces the direct sum and the tensor product operations. Similar to how a ring is preferred to a semiring, one single vector space is preferred to a pair of vector spaces V_1, and V_2. The direct sum $V_1 \oplus V_2$ provides the natural vector space that V_1 and V_2 sit inside. Similarly, the tensor product returns the natural vector space $V_1 \otimes V_2$ that the Cartesian product $V_1 \times V_2$ sits inside. The direct sum and tensor product can be applied to a variety of categories — abelian groups, commutative rings, and even vector bundles as addressed in the next chapter. To define the direct sum for these categories, this section considers a more general category, the category of modules. In defining the direct sum and tensor product operation through modules, the definitions for other categories will follow quickly.

Definition 2.4 (Module)**.** Let M be a set and let R be a commutative ring with identity. Further, take an additive operation $+ : M \times M \to M$ and a scalar multiplication from $R \times M$ to M. Then, M is a *module over* R if:

(i) M with $+$ forms an abelian group.
(ii) $(r+s)m = rm + sm$ for all $r, s \in R$ and $m \in M$
(iii) $r(m+n) = rm + rn$ for all $r \in R$ and $m, n \in M$
(iv) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$
(v) $1 \cdot m = m$ for all $m \in M$

Note that the properties of a module are exactly those properties of a vector space. The only difference between modules and vector spaces is that a vector space is over a field whereas a module can be over a ring. Modules then, are a generalization of vector spaces.

Additionally, consider any module over the ring $\mathbb{Z}$ where scalar multiplication $\mathbb{Z} \times M \to M$ is defined by $(n, m) \mapsto m + m + \cdots + m$ where the addition is performed n times. This indeed satisfies the properties of a module, but this definition makes the structure given by scalar multiplication redundant — it is only a shorthand for repeated addition. Then, this module is simply an abelian group. In fact, $\mathbb{Z}$ modules and abelian groups are exactly equivalent. Commutative rings are not modules, but giving a $\mathbb{Z}$ module M an appropriate multiplication operation $M \times M \to M$ could then make a module into a commutative ring. Overall, vector spaces, abelian groups, and commutative rings are modules with extra specifications.

Modules are a category. Keeping in mind that modules are generalizations of vector spaces, the natural homomorphism to associate with with modules is a linear map.

Definition 2.5 (Module Homomorphism)**.** Let R be a commutative ring and let M and N be R-modules. Then, a *homomorphism of modules* is a mapping $\varphi : M \to N$ such that

(i) $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$ for all $m_1, m_2 \in M$.
(ii) $\varphi(rm) = r\varphi(m)$ for all $r \in R$ and $m \in M$.

With the definition of the module category, note that the direct sum of two modules M_1 and M_2 is the natural single module that both M_1 and M_2 sit inside. This condition is precisely formulated with the universal property of direct sum as in the following definition.

Definition 2.6 (Direct Sum)**.** Take commutative ring R with identity and consider two R-modules M_1 and M_2. Then the *direct sum* of the modules, denoted $M_1 \oplus M_2$, is the unique R-module and injective inclusion maps $i_1 : M_1 \to M_1 \oplus M_2$ and $i_2 : M_2 \to M_1 \oplus M_2$ such that the universal property is satisfied. That is, for any R-module N with the pair of homomorphisms $\varphi_1 : M_1 \to N$ and $\varphi_2 : M_2 \to N$, there exists a unique homomorphism of R-modules $\psi : M_1 \oplus M_2 \to N$ such that the following diagram commutes. In words, this is the requirement that $\varphi_1 = \psi \circ i_1$ and $\varphi_2 = \psi \circ i_2$.

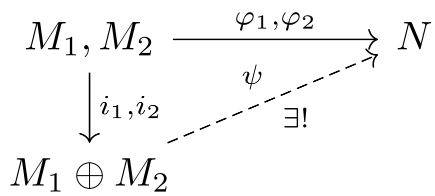

FIGURE 3. Universal Property of Direct Sum

Uniqueness of the direct sum follows directly from the universal property as mentioned in the ring completion section. However, existence must be shown by an explicit construction. In the finite case $M_1 \oplus M_2$, the construction is simply the set $M_1 \times M_2$ where the operations are defined coordinate-wise. This construction together with the inclusion maps $i_1 : m_1 \mapsto (m_1, 0)$ and $i_2 : m_2 \mapsto (0, m_2)$ for $m_1 \in M_1$, $m_2 \in M_2$ satisfies the universal property and thus gives the direct sum. For infinite direct sums, however, a valid construction can only allow a finite number of coordinates to be nonzero, which is the key distinction between direct sum and Cartesian product.

Recall the categories of vector spaces, abelian groups, and commutative rings are all modules with additional structure. Thus this definition of direct sum between modules defines the direct sum on each of these categories. However, borrowing thus module operation only promises that the resulting direct sum will be a module — not a vector space, abelian group, or commutative ring. It must be shown that the additional structure for each category is preserved in some way.

The verification for vector spaces, for example, is brief but should be emphasized. Take two vector spaces with field F; these two vector spaces are F-modules, thus the direct sum gives an F-module, which is in turn a vector space. Similarly, two abelian groups are $\mathbb{Z}$ modules, and so the direct sum is a $\mathbb{Z}$-module which is in turn an abelian group.

However, showing that the direct sum of commutative rings results in a commutative ring takes more work to verify, for there is no predefined multiplication mapping on the direct sum. To define this multiplication map, note the convention of denoting $i_1(m_1)+i_2(m_2)$ for $m_1 \in M_1$ and $m_2 \in M_2$ by $m_1 \oplus m_2$. Now, consider rings R and S and let the multiplication map in the direct sum $R \oplus S$ be defined by $(r_1 \oplus s_1) \cdot (r_2 \oplus s_2) = (r_1 s_1 \oplus r_2 s_2)$. This indeed defines a well-defined multiplication map in the direct sum that satisfies necessary properties to make $R \oplus S$ into a commutative rings. However, there is no natural choice for the identity in $R \oplus S$ that will agree with the inclusion maps, so the direct sum of commutative rings will be a commutative ring, but not necessarily with identity.

Example 2.7. This example claims that the direct sum of vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ is isomorphic to $\mathbb{R}^{n+m}$ with some natural inclusion maps. Take the inclusion maps to be given by $i_1 : \mathbb{R}^n \to \mathbb{R}^{n+m}$

and $i_2 : \mathbb{R}^m \to \mathbb{R}^{n+m}$ defined by

$$i_1 : (v_1, v_2, \ldots, v_n) \mapsto (v_1, v_2, \ldots, v_n, 0, 0, \ldots, 0)$$
$$i_2 : (w_1, w_2, \ldots, w_m) \mapsto (0, 0, \ldots, 0, w_1, w_2, \ldots, w_m)$$

By the uniqueness of direct sum, it must only be verified that $\mathbb{R}^{n+m}$ satisfies the defining universal property of direct sum. So take V to be any vector space and take linear maps $\varphi_1 : \mathbb{R}^n \to V$ and $\varphi_2 : \mathbb{R}^m \to V$. Now consider the linear map $\psi : \mathbb{R}^{n+m} \to V$ given by

$$\psi : (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m) \mapsto \varphi_1(v_1, v_2, \ldots, v_n) + \varphi_2(w_1, w_2, \ldots, w_m)$$

This indeed satisfies the necessary properties $\varphi_1 = \psi \circ i_1$ and $\varphi_2 = \psi \circ i_2$. And the linearity of ψ follows from that of φ_1 and φ_2. This gives existence and for uniqueness note that the restrictions upon ψ force it to agree with the defined ψ on the basis vectors. Because a linear map defined on the basis vectors extends to a unique map, only the defined map ψ can work, completing the verification. In the same way it follows that $\mathbb{C}^n \oplus \mathbb{C}^m \cong \mathbb{C}^{n+m}$.

The direct sum can be extended to more than two modules by initially considering a collection of more than two modules and corresponding maps in the universal property. In fact, consider a collection M_λ where λ is indexed by some index set I. Then using a collection of homomorphisms φ_λ in the universal property indexed by the same set I gives the direct sum $\bigoplus_{\lambda \in I} M_\lambda$ of an arbitrarily large set.

Now note some properties of direct sum. Firstly, the direct sum is associative, for both the sum $M_1 \oplus (M_2 \oplus M_3)$ and the sum $(M_1 \oplus M_2) \oplus M_3$ are defined to be the direct sum $M_1 \oplus M_2 \oplus M_3$ as defined in the previous paragraph. Next note that commutativity only depends on if the initial pair of modules is denoted M_1, M_2 or M_2, M_1. This is just a notational question and so there is an isomorphism between $M_1 \oplus M_2$ and $M_2 \oplus M_1$, giving commutativity.

Additionally, the direct sum operation has an identity — the module $\{0\}$. The set containing 0 is also a vector space, an abelian group, and a commutative ring, so this acts an an identity operation for every category of interest. The verification that $\{0\}$ is the additive identity follows by the same process as example 2.7. It must only be shown that M together with inclusion maps fulfills the defining universal property of $M \oplus \{0\}$ and thus must be isomorphic by the uniqueness of direct sum.

Similar to the direct sum is the tensor product, which takes the Cartesian product of two modules and returns the module that the Cartesian product sits within. The following definition refers to *bilinear maps*, which is simply a map $\omega : M_1 \times M_2 \to N$ such that for each $m_1 \in M_1$ the maps $\omega_1 : x \mapsto \omega(x, m_2)$ is and $\omega_2 : x \mapsto \omega(m_1, x)$ are both linear.

Definition 2.8 (Tensor Product)**.** Take commutative ring R with identity and take M_1 and M_2 to be R-modules. Then, the *tensor product* of M_1 and M_2, denoted $M_1 \otimes M_2$ is the unique R-module together with an injective bilinear map $b : M_1 \times M_2 \to M_1 \otimes M_2$ such that the universal property is satisfied. That is, for for any R-module N with corresponding bilinear map $\omega : M_1 \times M_2 \to N$, there exists a unique homomorphism of modules $\psi : M_1 \otimes M_2 \to N$ such that the following diagram commutes. In words, $\psi \circ b = \omega$.

Again, uniqueness of the direct sum follows automatically from the universal property so only the existence needs to be verified with a construction, which will be ommited in this document. See [**2**] for a complete discussion on the tensor product.

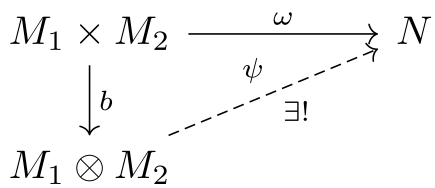

FIGURE 4. Universal Property of Tensor Products

Extending the tensor product to the categories of vector spaces, abelian groups, and commutative rings requires verifying that the resulting module has the necessary extra structure. The reasoning for vector spaces and abelian groups is identical to that for direct sum. However, extending this definition to rings again takes some work. First note the shorthand $b(m_1, m_2) = m_1 \otimes m_2$, and use this to define the multiplication operation between the tensor products of two rings R and S. Specifically, $(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) = (r_1 r_2 \otimes s_1 s_2)$. Of course, the linear map b has no promise of being surjective and so not every element in $S \otimes R$ is expressible as the "pure tensor" $r \otimes s$. However, every element is expressible as a linear combination of pure tensors and so the given definition of multiplication still extends to the tensor product as a whole. This definition of multiplication satisfies all of the necessary properties giving the tensor product of rings. In fact, if the original rings has identities 1_R and 1_S, then the element $1_R \otimes 1_S$ is an identity in the tensor product that agrees with the bilinear map.

Example 2.9. Again consider the vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. This example will demonstrate the isomorphism $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm}$ by showing $\mathbb{R}^{nm}$ satisfies the necessary universal property. For notational convenience, denote the elements of $\mathbb{R}^{nm}$ by $n \times m$ matrices. Then define the bilinear map $b : R^n \times \mathbb{R}^m \to \mathbb{R}^{nm}$ by

$$b : (\vec{v}, \vec{w}) \mapsto \vec{v}\vec{w}^T.$$

Where the above definition considers the matrix multiplication of the column vector $\vec{v}$ with the row vector $\vec{w}^T$. Next consider a vector space V with bilinear map $\omega : \mathbb{R}^n \times \mathbb{R}^m \to V$ and observe how the universal property mapping ψ must act on the basis vectors. Let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis for $\mathbb{R}^n$ and $\{f_1, f_2, \ldots, f_m\}$ denote the standard basis for $\mathbb{R}^m$. Then note that the set $\left\{e_i f_j^T\right\}$ letting i and j range gives a basis for $\mathbb{R}^{nm}$ where the matrix $e_i f_j^T$ has a 1 in the i, j entry and 0's elsewhere. Next, note that by the composition requirement, $\psi(e_i f_j^T) = \omega(e_i, f_j)$. But then this defines the mapping ψ on a basis and thus extends ψ to the following unique linear map.

$$A \mapsto \sum A_{ij}\omega(e_i, f_j).$$

This indeed satisfies the composition requirement completing the proof. Similarly, $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$.

Example 2.10. Consider the quotient rings $\mathbb{Z}[\alpha]/(\alpha^2)$ and $\mathbb{Z}[\beta]/(\beta^2)$. This example will show that $\mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta]/(\beta)^2 \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$. Again, it must only be shown that $\mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ satisfies the relevant universal property by the uniqueness of the universal property. Now define the bilinear map $b : \mathbb{Z}[\alpha]/(\alpha^2) \times \mathbb{Z}[\beta]/(\beta^2) \to \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ by

$$b : (n_1 + m_1\alpha, n_2 + m_2\beta) \mapsto (n_1 + m_1\alpha)(n_2 + m_2\beta)$$

By this choice of bilinear map, note that if a map ψ is to fulfill the universal property, it must satisfy $\psi(1) = \omega(1, 1)$, $\psi(\alpha) = \omega(\alpha, 1)$, $\psi(\beta) = \omega(1, \beta)$, and $\psi(\alpha\beta) = \omega(\alpha, \beta)$. This defines ψ on the additive basis, which forces ψ to have the following expression.

$$\psi : n + a\alpha + b\beta + m\alpha\beta \mapsto n \cdot \omega(1,1) + a \cdot \omega(\alpha, 1) + b \cdot \omega(1, \beta) + m \cdot \omega(\alpha, \beta)$$

This indeed satisfies the universal property, concluding the proof.

Next note some important properties of tensor product. The initial ordering of the modules is simply notational and so there is an isomorphism $M_1 \otimes M_2 \cong M_2 \otimes M_1$, giving commutativity of the tensor product operation. The tensor product of multiple modules is defined by considering some a Cartesian product $M_1 \times M_2 \times M_3$ together with *multilinear maps* that have multiple entries and are linear in each entry; with this definition, associativity is automatic.

An interesting property of tensor product is that $M \otimes R \cong M$ for an R-module M. The proof for this follows the same process as in Example 2.9. It must only be shown that M together with a bilinear map satisfies the defining universal property of tensor product and thus must be isomorphic to the tensor product by the universal property. Note that a real vector space is an $\mathbb{R}$-module, and so $\mathbb{R}^n \otimes \mathbb{R} \cong \mathbb{R}^n$, which agrees with 2.9. Further recall that abelian groups and commutative rings are considered as $\mathbb{Z}$-modules and so $G \otimes \mathbb{Z} \cong G$ for an abelian group G and $R \otimes \mathbb{Z} \cong R$ for a commutative ring R.

Tensor products and direct sums have an interesting relationship. In particular, for modules M_1, M_2, and M_3, it follows that $M_1 \otimes (M_2 \oplus M_3) \cong (M_1 \otimes M_2) \oplus (M_2 \otimes M_3)$. That is, tensor product distributes over direct sum. The verification for this is again in the same spirit as in Examples 2.7 and 2.9. That is, it must be shown that $M_1 \otimes (M_2 \oplus M_3)$, which satisfies the defining property of tensor product fulfills the defining universal property of direct sum for $(M_1 \otimes M_2) \oplus (M_2 \otimes M_3)$.

For convenience, all of the properties of direct sum and tensor product discussed previously are now summarized.

Claim 2.11. Let M, M_1, M_2, and M_3 be R-modules. However, the following holds for vector spaces, abelian groups, and commutative rings.

(i) Commutativity of direct sum: $M_1 \oplus M_2 \cong M_2 \oplus M_1$.
(ii) Associativity of direct sum: $M_1 \oplus (M_2 \oplus M_3) \cong (M_1 \oplus M_2) \oplus M_3$.
(iii) $\{0\}$ acts as an identity for direct sum: $M \oplus \{0\} \cong M$.
(iv) Commutativity of tensor product: $M_1 \otimes M_2 \cong M_2 \otimes M_1$.
(v) Associativity of tensor product: $M_1 \otimes (M_2 \otimes M_3) \cong (M_1 \otimes M_2) \otimes M_3$.
(vi) R acts as an identity for tensor product: $M \otimes R \cong R$.
(vii) Tensor product distributes over direct sum: $M_1 \otimes (M_2 \oplus M_3) \cong (M_1 \otimes M_2) \oplus (M_1 \otimes M_3)$.

All of these properties were discussed previously.

3. Verifications

Proof of Existence of Definition 2.3. The existence of a ring completion is shown through an explicit construction. Take any commutative semiring with additive cancellation $(S, +, \cdot)$ and consider the equivalence relation $\sim$ on $S \times S$ defined as follows: for (a_1, b_1), (a_2, b_2) in $S \times S$, then let $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. The aim is to make the set of equivalence classes under $\sim$ into a ring.

First, define the additive operation + by

$$[(a,b)]+[(c,d)]=[(a+c,b+d)]$$

Next, define the multiplicative operation $\cdot$ by

$$[(a,b)]\cdot[(c,d)]=[(ac+bd,ad+bc)]$$

This proof aims to verify that the set of equivalence classes $S\times S/\sim$ paired with the operations $(+,\cdot)$ forms a commutative ring that is a ring completion of S.

It must be verified that the additive operation is well defined, so consider elements (a_1,b_1), (a_2,b_2), (c_1,d_1), (c_2,d_2) in $S\times S$ such that $(a_1,b_1)\sim(a_2,b_2)$ and $(c_1,d_1)\sim(c_2,d_2)$. Then, I claim that $(a_1+c_1,b_1+d_1)\sim(a_2+c_2,b_2+d_2)$. Indeed, this satisfies the definition of the equivalence relation, for

$$\begin{aligned}&(a_1+c_1)+(b_2+d_2)=(a_1+b_2)+(c_1+d_2)\\&=(a_2+b_1)+(c_2+d_1)=(a_2+c_2)+(b_1+d_1)\end{aligned}$$

where the above computation used the substitutions $a_1+b_2=a_2+b_1$ and $c_1+d_2=c_2+d_1$ promised by the relations $(m_1,m_2)\sim(m_1',m_2')$ and $(l_1,l_2)\sim(l_1',l_2')$. This confirms that + is well-defined on $(S\times S)/\sim$.

The transitivity and commutativity of + on the equivalence classes follows immediately from the commutativity and transitivity of the operation + on S.

Next, note that the additive identity in $(S\times S)/\sim$ is given by $[(0,0)]$ where 0 denotes the identity element in S. Indeed, we have $[(a,b)]+[(0,0)]=[(a,b)]$ for any element $[(a,b)]$.

The proposed ring has an inverse mapping for the addition operation. Consider an element $[(a,b)]$. Then, I claim the element $[(b,a)]$ forms the desired inverse. To see this, consider the sum $[(a+b,b+a)]$ and note that $(a+b)+0=0+(b+a)$, which shows $[(a+b,b+a)]=[(0,0)]$.

It must be verified that the multiplicative operation is well-defined before verifying any further properties. Consider the elements $(a_1,b_1)\sim(a_2,b_2)$ and $(c_1,d_1)\sim(c_2,d_2)$ in $S\times S$. It then must be verified that $(a_1c_1+b_1d_1,a_1d_1+b_1c_1)\sim(a_2c_2+b_2d_2,a_2d_2+b_2c_2)$. To accomplish this, consider the following $M_1,M_2\in S$:

$$\begin{aligned}M_1&=c_2(a_1+b_1)+b_2(c_1+d_1)+b_2c_2\\M_2&=c_1(a_2+b_2)+b_1(c_2+d_2)+b_1c_1\end{aligned}$$

Next, observe that using the relations $a_1+b_2=a_2+b_1$ and $c_1+d_2=c_2+d_1$, it follows that $a_1c_1+b_1d_1+M_1=a_2c_2+b_2d_2+M_2$.

$$\begin{aligned}&a_1c_1+b_1d_1+M_1=a_1c_1+b_1d_1+c_2a_1+c_2b_1+b_2c_1+b_2d_1+b_2c_2\\&=(a_1+b_2)(c_1+c_2)+(d_1+c_2)(b_1+b_2)\\&=(a_2+b_1)(c_1+c_2)+(d_2+c_1)(b_1+b_2)\\&=a_2c_2+b_2d_2+c_1a_2+c_1b_2+b_1c_2+b_1d_2+b_1c_1=a_2c_2+b_2d_2+M_2\end{aligned}$$

A similar process shows that $a_1d_1+b_1c_1+M_1=a_2d_2+b_2c_2+M_2$. Then, summing the two results gives

$$(a_1c_1+b_1d_1)+(a_2d_2+b_2c_2)+(M_1+M_2)=(a_2c_2+b_2d_2)+(a_1d_1+b_1c_1)+(M_1+M_2)$$

Applying the additive cancellation property of S to the term $(M_1 + M_2)$ gives the desired relation and provides the conclusion $(a_1c_1 + b_1d_1, a_1d_1 + b_1c_1) \sim (a_2c_2 + b_2d_2, a_2d_2 + b_2c_2)$ and so the multiplicative operation is well defined.

The transitivity of the multiplicative operation follows directly from $+$ and $\cdot$ transitive in S. Similarly, the commutativity of the multiplicative operation follows directly from the commutativity of $+$ and $\cdot$ in S.

Next, note that the element $[(1, 0)]$ acts as an identity element for the multiplicative operation. Indeed, $[(1, 0)] \cdot [(a, b)] = [(a, b)]$ for any element $[(a, b)]$.

It only remains to show that $+$ distributes over $\cdot$ to verify that $S \times S/\sim$ forms a ring. Indeed, for elements $[(a, b)]$, $[(c, d)]$, $[(e, f)]$:

$$\begin{aligned}
&[(e, f)] \cdot ([(a, b)] + [(c, d)]) = [(e, f)] \cdot [(a + c, b + d)] \\
&= [(ea + fb + ec + fd, eb + ed + fa + fe)] \\
&= [(ea + fb, eb + fa)] + [(ec + fd, ed + fc)] = [(e, f)] \cdot [(a, b)] + [(e, f)] \cdot [(e, d)]
\end{aligned}$$

Thus we have that $(S \times S)/\sim$ forms a commutative ring under the proposed operations. However, it remains to show that $(S \times S)/\sim$ is a valid ring completion. The necessary inclusion map $i : S \to (S \times S)/\sim$ is given by $i(s) = [(s, 0)]$. Then, take any ring R' and homomorphism $\varphi : S \to R'$; the existence and uniqueness of a commuting ring homomorphism $\psi : (S \times S)/\sim \to R'$ must be shown.

Uniqueness follows quickly from its homomorphism properties and the commutativity of the universal property. Indeed, take two commuting ring homomorphisms ψ and ψ' from $S \times S/\sim$ to R'. Then, the restrictions $\psi \circ i = \varphi$ and $\psi' \circ i = \varphi$ paired with i injective gives that $\psi = \psi'$ over the image $i(S)$. Then observe that any element $[(a, b)]$ is the composition of elements in $i(S)$ by $[(a, b)] = [(a, 0)] - [(b, 0)]$. Then, the homomorphism properties of rings extends ψ and ψ' to be equivalent over all of $(S \times S)/\sim$ giving uniqueness.

It only remains to show existence of the homomorphism. The map $\psi : [(a, b)] \mapsto \varphi(a) - \varphi(b)$ works. Commutativity follows easily, for $(\psi \circ i)(s) = \psi([(s, 0)]) = \varphi(s)$ for all $s \in S$. Now, it must be verified that ψ is a homomorphism. So, consider elements $[(a, b)]$ and $[(c, d)]$ of the ring completion.

The following equality chain shows that the the additive property of φ gives the additive property of ψ.

$$\begin{aligned}
&\psi([(a, b) + (c, d)]) = \psi([(a + c, b + d)]) = \varphi(a + c) - \varphi(b + d) \\
&= (\varphi(a) - \varphi(b)) + (\varphi(c) - \varphi(d)) = \psi([(a, b)]) + \psi([(c, d)])
\end{aligned}$$

Similarly, the additive and multiplicative property of φ gives the multiplicative property of ψ.

$$\begin{aligned}
&\psi([(a, b)] \cdot [(c, d)]) = \psi([(ac + bd, ad + bc)]) \\
&= \varphi(ac + bd) - \varphi(ad + bc) = \varphi(a)\varphi(c) + \varphi(b)\varphi(d) - \varphi(b)\varphi(c) - \varphi(a)\varphi(d) \\
&= (\varphi(a) - \varphi(b))(\varphi(c) - \varphi(d)) = \psi([(a, b)]) \cdot \psi([(c, d)])
\end{aligned}$$

Finally $\psi(1) = \psi([(1, 0)]) = \varphi(1) = 1$, completing the proof.

□

CHAPTER 3

Topology

1. The Category of Topological Spaces

Recall that a metric space is simply a set paired with a way of telling distance between two points and that a metric spaces gives rise to a notion of open sets. To begin this section, consider the following instance of a metric space.

Example 3.1 (The Interval as a Metric Space)**.** Consider the metric space that takes the interval $[0, 1]$ as a set and define the distance between two points $x, y \in [0, 1]$ to be $|x - y|$. This gives a notion of *open sets* on the interval by declaring a set U open if for every point x in the set there is some distance ε such that all points of distance less than ε from x are contained in U. For instance the open interval $(1/3, 2/3)$ would be declared open.

This section addresses a new category — the category of topological spaces. Topology looks at the open sets that come as a consequence of a metric and asks and asks: what if the *only* known thing about a space is what the open sets are. What information does this give about the space?

Common terminology hints at this central role of open sets in topology. A set *has a topology* if it is known what subsets are open and a set is *topologized* through the declaration of the open subsets. Below is a formal definition of a topological space. Keep in mind a topological space is simply a set together with a collection of open sets, often denoted $\mathcal{T}$, that follows some rules.

Definition 3.2 (Topological Space)**.** Given a set X and $\mathcal{T} \subseteq \mathcal{P}(X)$, the pair $(X, \mathcal{T})$ is called a *topological space* if:

(i) The empty set $\emptyset$ and the full set X are in $\mathcal{T}$.
(ii) For every set $\mathcal{U}_\alpha$ in $\mathcal{T}$ where $\alpha \in I$, the infinite union $\cup_{\alpha \in I} \mathcal{U}_\alpha$ is in $\mathcal{T}$.
(iii) For any two subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathcal{T}$, the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is in $\mathcal{T}$.

For $\mathcal{U} \in \mathcal{T}$, the element $\mathcal{U}$ is called an *open* set and its complement $\overline{\mathcal{U}}$ is called a *closed set*.

Compare the above definition to the open sets in a metric space. When the open sets are given by a metric, it follows that the infinite union of open sets is open and the finite intersection of open sets is open. Topology focuses more directly on the open sets themselves and so makes these union and intersection properties the defining characteristics of open sets. Topology describes the same structure as metric spaces, but from a different point starting point. It should be emphasized, however, that topology is more general than the notion of a metric space. While all metric spaces are topological spaces, not all topological spaces can be given a valid metric. Below is the same set $[0, 1]$ defined earlier, but now considered as a topological space.

Example 3.3 (The Interval as a Topological Space)**.** Consider the topological space by considering the set $[0, 1]$. Let the open sets $\mathcal{T}$ on the topological space be the smallest[1] collection of sets $\mathcal{T}$ such that each open interval (a, b) is in $\mathcal{T}$ and that $\mathcal{T}$ defines a valid topology on the interval. This defines the same open sets that result from Example 3.1 by using the standard metric[2], but from a purely topological point of view. The interval is denoted by I and understood to represent the set $[0, 1]$ wit the standard topology.

The topology on a space X that results from the standard metric as in the above example is called the *standard topology* on X.

The category of topological spaces is still incomplete — topological objects require some notion of morphisms between them. The inspiration for a good choice of morphism comes from the notion of continuous maps between metric spaces. In metric spaces, one equivalent way to define continuity is through open sets. This definition fits nicely in topology, so the category of topology borrows this definition to define continuity between topological spaces, which will be the morphisms of the category.

Definition 3.4 (Continuous Function)**.** Take two topological spaces and corresponding open sets $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. A function between the spaces $f : X \to Y$ is called *continuous* if for every open set $V \in \mathcal{T}_Y$, its inverse image is open: $f^{-1}(V) \in \mathcal{T}_X$.

As an example of a continuous function, consider the following mapping between intervals.

Example 3.5. As an example of a continuous function, consider the interval I with the standard topology and let the function $f : I \to I$ be given by $f(x) = 0$. To see that this map is continuous, take any set V in the codomain. Consider two cases: $0 \in V$ and $0 \notin V$. In the $0 \in V$ case, then the inverse image is the entire domain, that is $f^{-1}(V) = I$, and the domain as a whole is open. In the case that $0 \notin V$, then the inverse image is the empty set, in other words $f^{-1}(V) = \emptyset$, and the empty set is open. Thus f is continuous.

Note that the above example does not actually depend on the topology of I. This unusual; in fact, the topology typically has a large impact on whether a function is continuous.

Example 3.6. Take two topological objects $(I_1, \mathcal{T}_1)$ and $(I_2, \mathcal{T}_2)$ where I_1 and I_2 denote the interval and $\mathcal{T}_1$ and $\mathcal{T}_2$ will be discussed later. Then, let $f : I_1 \to I_2$ be a function given by $f(x) = x$. Now consider two possibilities:

- Firstly, take $\mathcal{T}_1$ to be the power set $\mathcal{P}(I_1)$ and take $\mathcal{T}_2 = \{\emptyset, I_2\}$. Then, the preimage of every set V in I_2 is indeed open, for every set in I_1 is open and thus f is continuous.
- Next, swap the topologies. Take $\mathcal{T}_1 = \{\emptyset, I_1\}$ and $\mathcal{T}_2 = \mathcal{P}(I_1)$. Then, many open sets have a non-open preimage. For instance, the set $\{0\} \subset \mathcal{T}_2$ is open in this topology, but the preimage is given by $\{0\} \subset I_1$, which is not one of the two open sets I_1. Thus this map is not continuous.

As hinted at before, the category of topological spaces is similar to the category of metric spaces. Given a metric space, the objects can be translated to topological objects and the continuous functions between metric spaces can interpreted as continuous functions between topological objects.

[1] $\mathcal{T}_1$ is smaller than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$

[2] A standard metric measures distance by Euclidean standards as in the real world

This idea is technically a functor from the category of metric spaces to the category of topological spaces. This functor can be used to define topological objects as in the following example.

Example 3.7 (S^2)**.** Consider the unit sphere as understood as a metric space. That is, take the set $X \subset \mathbb{R}^3$ defined by $X = \{(x, y, z) \in \mathbb{R} | x^2 + y^2 + z^2 = 1\}$. Then, consider the standard metric d in $\mathbb{R}^3$ and the pair (X, d) defines the unit sphere as a metric space. By considering all open sets $\mathcal{T}$ as defined by the metric d, the pair (X, d) then defines the topology of the two dimensional sphere S^2. Take particular note that this process is due to the functor from metric spaces to topological spaces.

This process of defining topological spaces through metric spaces provides many fundamental topological objects that we can build upon. In particular, the topology of an interval I, the topology of any sphere S^n, and the topology of $\mathbb{R}^n$ itself arise by using the standard metric of $\mathbb{R}^n$.

Recall that an isomorphism as generally defined in category theory is a morphism that has an inverse morphism, suggesting the following definition in topology.

Definition 3.8 (Isomorphism)**.** A function f between topological spaces is an *isomorphism* if f is continuous with a continuous inverse. An isomorphism of topological spaces is denoted by $\approx$.

An isomorphism in topology corresponds to a mapping that takes one space to another without removing or adding connections between the space. Consider the following simple example of an isomorphism.

Example 3.9. Take the spaces $[0, 1]$ and $[0, 2]$ equipped with the standard topology as defined by the standard metric. Then the mapping $f : [0, 1] \to [0, 2]$ defined by $f : x \mapsto 2x$ is an isomorphism. Indeed, for any open set $U \subset [0, 2]$, each point y has a distance ε_y such that all points within distance ε_y from y are contained in U. The preimage $f^{-1}(U)$ is exactly U with each point y relabeled by $y/2$, so considering the distance $\varepsilon_y/2$ for each point y demonstrates that $f^{-1}(U)$ is open, giving continuity of f. A similar argument applies to the inverse $f^{-1} : y \mapsto y/2$, thus f is an isomorphism.

2. Some Mappings and Corresponding Topologies

This section addresses three useful mappings in topology: the quotient map, the inclusion map, and the projection map. Ideally, these three maps will be continuous functions so that they are morphisms of the category. However, as demonstrated in example 3.6, changing the topology can *make* a function continuous. Each of these three mappings has a corresponding topology that provides the desired continuity. With the completion of this section, there will be no need to stress about whether a quotient map, inclusion map, or projection map is continuous — the assumed topology will make it continuous.

A quotient map is a map from a topological X to a quotient $X/\sim$ by some equivalence relation $\sim$ on X. Let Y denote the set of equivalence classes $X/\sim$. Because elements of Y are subsets of X, there is a natural choice of topology $\mathcal{T}_Y$. For a single equivalence class $[x] \in Y$, the natural choice is to say $[x] \in \mathcal{T}_Y$ exactly when $[x] \in \mathcal{T}_X$. In general, any group of equivalence classes $\cup_\alpha [x_\alpha]$ in Y should be open in Y exactly when $\cup_\alpha [x_\alpha]$ is open in X. This topology makes the mapping $x \mapsto [x]$ continuous. However, the quotient map is a better starting point to define the quotient topology, but note that the following definition is inspired by the above.

Definition 3.10 (Quotient Map and Quotient Topology)**.** Consider topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ along with a surjective function $q : X \to Y$. Further restrict q such that $V \in \mathcal{T}_Y$ if and only if $q^{-1}(V) \in \mathcal{T}_X$. Then, q is a *quotient map.* Note that in the case that $\mathcal{T}_Y$ is not defined, a specified quotient map defines a *quotient topology* on Y.

Note that q requires exactly the sets such that it is always continuous. The condition $V \in \mathcal{T}_Y$ if and only if $q^{-1}(V) \in \mathcal{T}_X$ is analogous to the condition $\cup_\alpha [x_\alpha] \in T_Y$ if and only if $\cup_\alpha [x_\alpha] \in \mathcal{T}_X$ discussed previously. However, the quotient map is a more natural starting point and gives the equivalence relation immediately by $x_0 \sim x_1$ if and only if $q(x_0) = q(x_1)$. The following example defines a commonly used quotient map.

Example 3.11. An example of a quotient map that is widely used begins by considering a topological space X and a subspace $A \subset X$. Then apply the equivalence relation $\sim$ defined by $x \sim x$ and $a_1 \sim a_2$ if $a_1, a_2 \in A$. This defines a quotient map and corresponding quotient object $X/\sim$, which is often denoted X/A. This quotient map collapses the set $A \subset X$ to a point.

Many interesting topological objects arise from taking quotient maps.

Example 3.12 ($\mathbb{R}P^1 \approx S^1$)**.** The real projective line, denoted $\mathbb{R}P^1$, is defined by considering the equivalence relation $\sim$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ where $v_1 \sim v_2$ if $v_1 = \lambda v_2$ for some $\lambda \neq 0$. Then, the real projective line is the corresponding quotient space:

$$\mathbb{R}P^1 = (\mathbb{R}^2 \setminus \{(0,0)\})/\sim$$

The points making up half a unit circle are a set of representatives for $\mathbb{R}P^1$, but the nature of the equivalence relation makes it so that the oposite ends of this half circle are “connected”, suggesting $\mathbb{R}P^1 \approx S^1$. An explicit isomorphism $\varphi : \mathbb{R}P^1 \mapsto S^1$ motivated by the complex map $z \mapsto z^2$ is given by

$$[\vec{v}] \mapsto \frac{1}{\|v\|^2} \begin{pmatrix} v_x - v_y \\ 2v_x v_y \end{pmatrix}$$

Real projective space $\mathbb{R}P^n$ in general is constructed by applying the same equivalence relation on $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$. This can be further extended to complex projective space as in the following example, which gives a property of the sphere key to K-Theory.

Example 3.13 ($\mathbb{C}P^1 \approx S^2$)**.** Similar to the previous example, the complex projective line is denoted $\mathbb{C}P^1$ and is given the equivalence relation $\sim$ on $\mathbb{C}^2 \setminus \{(0,0)\}$ given by $v_1 \sim v_2$ if $v_1 = \lambda v_2$ for some $\lambda \neq 0$, but λ is now allowed to be complex. Then the complex projective line is then the corresponding quotient space:

$$\mathbb{C}P^1 = (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$$

The space $\mathbb{C}P^1$ is defined using a four dimensional space and so is difficult to visualize. However, note the isomorphism $\mathbb{C}P^1 \approx S^2$. The bottom line for the justification involves the two injective maps $\varphi_1 : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}P^1$ and $\varphi_{-1} : S^2 \setminus \{(0,0,-1)\} \to \mathbb{C}P^1$ defined by

$$\varphi_1 : (x,y,z) \mapsto \left[\begin{pmatrix} \frac{x+iy}{1-z} \\ 1 \end{pmatrix}\right] \quad \text{and} \quad \varphi_{-1} : (x,y,z) \mapsto \left[\begin{pmatrix} 1 \\ \frac{x-iy}{1+z} \end{pmatrix}\right]$$

It follows that $\varphi_1(x,y,z) = \varphi_{-1}(x,y,z)$ over their shared domain by remembering $x^2+y^2+z^2 = 1$. The image of these two maps together is all of $\mathbb{C}P^1$ and so considering these two maps together provides an isomorphism $S^2 \to \mathbb{C}P^1$. For the interested reader, the idea behind the maps φ_1

and φ_{-1} is the combination of homogeneous coordinates for projective space and stereographic projection for the sphere.

Along with the quotient topology and corresponding quotient map, there is a subspace topology and corresponding inclusion map. Consider a topological space $(X, \mathcal{T}_X)$ and an open subset $A \subset X$ with no pre-defined topology. However, the subset A can borrow topology from the larger space X in a natural way: define $\mathcal{T}_A$ by $\mathcal{U} \in \mathcal{T}_A$ exactly when $U \in T_X$. However, note that this definition only works when A is open in X because for A to be a topological space, the whole space must be open. A construction that works when A is not open is given by $\mathcal{T}_A = \{\mathcal{U} \cap A | \mathcal{U} \in \mathcal{T}_X\}$. This gives the same open sets as previously discussed for A open, but also defines a valid topology for A closed. This topology is constructed exactly so that the inclusion map $i : A \to X$ given by $i(a) = a$ is continuous. However, defining the inclusion map itself serves as a better starting point, but the following definition is motivated by the above.

Definition 3.14 (Inclusion Map and Subspace Topology)**.** Take topological spaces $(A, \mathcal{T}_A)$ and $(X, \mathcal{T}_X)$. Then, an injective map $i : A \to X$ is called an *inclusion map* when $\mathcal{U} \in \mathcal{T}_A$ if and only if there exists a $V \in \mathcal{T}_X$ such that $V \cap i(A) = i(\mathcal{U})$. In the case that $\mathcal{T}_A$ is not defined, a specified inclusion map defines a *subspace topology* on A.

The defined subspace topology is equivalent to the topology discussed previously, and note this construction is perfectly engineered so that the inclusion map is continuous. Take an inclusion map $i : A \to X$ and an open set $V \subset X$. The parts of X that do not contain the image of A is irrelevant to the inverse image, so $i^{-1}(V)$ is equivalent to $i^{-1}(i(A) \cap V)$. Then, by definition, this corresponds to an open subset $\mathcal{U}$ in A, giving continuity.

Example 3.15. Take the real line $\mathbb{R}$ and the subspace $[0, 1] \subset \mathbb{R}$ both equipped with the standard topology. $[0, 1]$ is a natural subspace of $\mathbb{R}$, so there is an inclusion $i : [0, 1] \to \mathbb{R}$. However, note that sets such as $[0, 1]$ and $[0, 1/2)$ are open in $[0, 1]$ but are not open in the larger space $\mathbb{R}$. This discrepancy is accounted for by the intersection with $i(A)$ in definition 3.14.

So, the quotient map is paired with the quotient topology and the inclusion map is paired with the subspace topology. Now consider the projection map, which is paired with the product topology.

Definition 3.16 (Projection)**.** Let $X_1, \dots, X_n$ be spaces with topologies $\mathcal{T}_1, \dots, \mathcal{T}_n$. Then the *projection onto the k^{th} factor* is given by the following.

$$p : X_1 \times \cdots \times X_k \times \cdots \times X_N \to X_K$$
$$p : (x_1, \dots, x_k, \dots, x_n) \mapsto x_k$$

However, whether such a projection is continuous depends on the choice of topology for the product $X_1 \times \cdots \times X_n$. The natural choice for this topology is the product topology, which is best defined through a universal property.

Definition 3.17 (Product Topology)**.** Let $X_1, \dots, X_n$ be spaces with topologies $\mathcal{T}_1, \dots, \mathcal{T}_n$. Then, consider the family of projections onto the k^{th} factor $p_k : X_1, \dots, X_n \to X_k$. Then, the product topology $\mathcal{T}_\times$ is the unique topology that satisfies the following universal property of product topology. That is, the product topology is such that each p_k is continuous, and for any topological space Y together with a family of continuous map $f_k : Y \to X_k$ there exists a unique continuous map

$f : Y \to X_1 \times \cdots \times X_n$ such that the following diagram commutes for each k. That is, $f_k = p_k \circ f$ for each k.

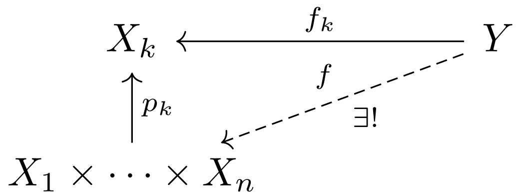

FIGURE 1. Universal Property of Product Topology

This definition still requires verifications of existence and uniqueness. Uniqueness follows automatically from the universal property; the proof follows similarly here even though the arrows are pointing in the opposite directions. However, the existence of a product topology must be verified through an explicit construction.

The bottom line with the product topology is that the universal defines the topology of $X_1 \times \cdots \times X_n$ exactly so that each of the projections p_k is continuous. When assuming the product topology, there is no need to stress about whether p_k is continuous — it will be just as the inclusion and quotient maps will be.

3. Nice Properties of Topological Objects

Topology does not assume a lot — only any set and a notion of open sets. This results in there being a lot of possible topological objects. Some of these objects are intuitive and have nice properties, but many of these objects are unintuitive and often difficult to work with. This section will aim to identify two nice properties — Hausdorff and compact. Restricting spaces to having these two nice properties will throw out the troublesome topological spaces that would obstruct the remainder of the story.

First note a nice property of the topological space $\mathbb{R}^2$ with the standard topology (which is defined by the standard metric). For any two distinct points $p, q \in \mathbb{R}^2$, it is possible to draw two tiny open sets — one surrounding around p and one surrounding q — such that the two open sets do not intersect. This corresponds to p and q having some distance between them — some "space" to themselves.

However, not all topological objects have this property. For instance, consider the same set $\mathbb{R}^2$ with a ridiculous metric: the distance between any two points is 0. The visual here is that all of $\mathbb{R}^2$ has been squashed into a single point. This then results in a ridiculous topology: $\mathcal{T} = \{\emptyset, \mathbb{R}^2\}$, containing only the empty set and the full set with nothing else. In this case, given any two points p and q it is impossible to draw two disjoint open sets where one contains p and the other contains q. The reason for this is that there is literally no distance between p and q. The best way to avoid working with this topological object is by imposing the Hausdorff condition. The definition follows, but intuitively think of the Hausdorff condition as requiring that any two distinct points have a nonzero distance between them.

Definition 3.18 (Hausdorff)**.** Let X be a topological space. We call X *Hausdorff* if for any pair of distinct points $p, q \in X(p \neq q)$, there exists open sets U_p with $p \in U_p$ and U_q with $q \in U_q$ such that $U_p \cap U_q = \emptyset$.

The Hausdorff property is required for the story to move forward. Another necessary property is *compactness*, which prevents the topological spaces from getting too "big". For example, consider the interval I with the standard metric in comparison to $\mathbb{R}$ with the standard metric. The space $\mathbb{R}$ is unbounded and in some sense bigger than I. The following definition characterizes this difference using only open sets.

Definition 3.19 (Compact)**.** A topological space X is *compact* if every open cover of X has a finite subcover. That is, for any cover $X = \cup_{\alpha \in I} U_\alpha$ with open $U_\alpha \subset X$ for all $\alpha \in I$, then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $X = U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots U_{\alpha_n}$.

With this criterion, spaces such as D^n and S^n are compact and spaces such as $\mathbb{R}^n$ are not compact. To practice, consider the following verification that $\mathbb{R}$ is not compact.

Example 3.20 ($\mathbb{R}^2$ is not compact)**.** In order to show $\mathbb{R}$ is not compact, it suffices to bring up a specific infinite open cover and prove it does not have a finite subcover. So consider the following infinite cover:

$$C = \{(k-1, k+1) \times \mathbb{R} : k \in \mathbb{Z}\}$$

However, consider removing any element $(k-1, k+1)$ from the set. Then, the resulting set $C \setminus \{(k-1, k+1)\}$ does not contain the point k in any of its sets and so it does not cover all of $\mathbb{R}$. Then, there is no way to reduce the cover and thus there is no finite subcover.

An additional nice property of topological spaces is for the topological space to have a particular point within the topological space. Often times operations on topological spaces require choosing an arbitrary point within the space and so having a particular point x in a topological space X in mind is useful. All such pairs (X, x) is an important category.

Definition 3.21 (Pointed Topological Space)**.** The category of pointed topological spaces has as objects all pairs (X, x) such that X is a topological space and $x \in X$. The morphisms between two pointed topological spaces (X, x) and (Y, y) is a continuous function $f : X \to Y$ with the additional restriction that $f(x) = f(y)$.

The operations wedge sum and smash product are discussed in the following section and rely on having a particular point within a topological space.

4. Operations on Topological Spaces

The cone operation C takes some topological space and expands it by making it into a cone. Before giving the formal definition, here is an illustration. Begin with S^0 — the set of two points. Then, as is illustrates in figure **??**, $C(S^0)$ would be create a low dimensional cone shape which is isomorphic to the interval I. Then, $C(I)$ would create a cone, which is isomorphic to the two dimensional disk D^2. If C^n denotes applying the cone operation n times, then $C^n(S^0) \approx D^n$ in general.

The formal definition of the cone operation combines the product and quotient topologies.

Definition 3.22 (Cone Operation)**.** Take a topological space X. Then, taking the interval $I = [0, 1]$, the cone operation C is given by

$$C(X) = (X \times I)/(X \times \{0\})$$

Note the quotient of a topological space is given in example **??**. Intuitively, this cone operation takes extends the topological space into a new dimension by crossing with the interval and then "pinches" the top.

Closely related to the cone operation is the suspension operation S. If the cone "cone-ifies" a topological space, the suspension operation "sphere-ifies" the topological space. Again, before giving the formal definition, consider an example sequence of suspension operations. Following Figure **??** and beginning with S^0, then $S(S^0)$ gives the circle S^1 and applying the suspension again gives $S(S^1)$ gives the sphere S^2. Overall, if S^n denotes applying the suspension operation n times, then $S^n(S^0) \approx S^n$ in general.

The formal definition of the suspension follows uses a similar construction as the cone operation.

Definition 3.23 (Suspension Operation)**.** Take a topological space X. Then, taking the interval $I = [0, 1]$, the suspension operation S is given by

$$S(X) = (X \times I)/(X \times \{0, 1\})$$

And so the suspension operation takes the Cartesian product with the interval and "pinches" two sides together. The wedge sum aims to glue to spaces together at a point. To avoid an arbitrary choice of point, only consider pointed topological spaces as defined previously.

Definition 3.24 (Wedge Sum)**.** Take two pointed topological spaces (X, x) and (Y, y). Further take the equivalence relation $\sim$ be defined by $x \sim y$ and every point relates to itself. Then the *wedge sum*, denoted $X \vee Y$ is the pointed topological space

$$X \vee Y = X \cup^* Y/\sim$$

Where $\cup^*$ denoted the disjoint union and the new point in the pointed topological space is chosen to be $x \sim y$.

Example 3.25. Consider the topological space S^1 with some point $x \in S^1$. The wedge sum $S^1 \vee S^1$ then glues the points together and results in a figure eight shape.

Related to the wedge sum is the smash product. To motivate the smash product, note that the wedge sum considers two pointed topological spaces and aims to return a pointed topological space — similar to the idea of direct sum. The smash product is then related to the idea of tensor product in that it takes the Cartesian product of two pointed topological spaces and applies a quotient map with the goal of providing a pointed topological space.

Definition 3.26 (Smash Product)**.** Take two pointed topological spaces (X, x) and (Y, y). Then the smash product $X \wedge Y$ is given by

$$X \wedge Y = X \times Y/(X \vee Y)$$

Where $X \vee Y$ denotes $(X \times y) \cup (x \times Y)$ in the Cartesian product.

Note that the quotient map condenses every point containing either x or y in the quotient map into a single point, so choosing this point allows the result to be a pointed topological space.

Example 3.27. Again consider the pointed topological space S^1 with some point $x \in S^1$. The smash product $S^1 \wedge S^1$ can be visualized as follows. Consider the torus $S^1 \times S^1$. Then $x \times S^1$ corresponds to a circle around the outside edge of the torus whereas $S^1 \times x$ corresponds to a circle through the hole of the torus. The union of these is exactly $S^1 \vee S^1$ and so $S^1 \times S^1/(S^1 \vee S^1)$ allows this figure eight shape to collapse to a point.

Example 3.28. Consider $S^1 \wedge I = (S^1 \times I)/(S^1 \wedge I)$. This space can be rewritten by the quotient

$$(I \times I)/((S^1 \times \{0,1\}) \cup (\{x_0\} \times I))$$

for the pre-chosen point $x_0 \in S^1$. To see this, note that the quotient by $(S^1 \times \{0,1\})$ glues the space $I \times I$ into a cylinder and sets the interval I to a point. The quotient $\{x_0\} \times I$ then sets S^1 to a point, so this is identical to the defining expression of $S^1 \wedge I$.

By the same reasoning, this conclusion holds more generally. That is, $S^1 \wedge X$ is equivalent to the space $I \times X$ quotiented by the subspace $(X \times \{0,1\}) \cup (\{x_0\} \times I)$. This is simply a stronger version of the quotient $SX = (I \times X)/(I \times \{0,1\})$, thus there is a quotient map $q : SX \to S^1 \wedge X$. This gives the more general quotient map $q : S^n X \to S^n \wedge X$. More detail of this construction is given in [**3**, p. 12] under "reduced suspension" and in [**4**, p. 54].

Example 3.29. Take a topological space X with $A \subset X$. Further, for some space Y containing A and X, denote $Y \cup CA$ to be the union of the two spaces by identifying each point in $A \subset Y$ with the corresponding point in $A \subset CA$. In the same way, denote $Y \cup CX$ to be the union by gluing each point in $A \subset Y$ onto the corresponding point in $A \subset CA$. With this notation, observe some patterns.

$$(X \cup CA)/CA \approx X/A$$

To see this, first recall that the quotient sets the subspace X/A to a point. Now note that $X \cap CA$ is only A, so within X, this quotient will only collapse the subspace A to a point. Thus, rewrite this quotient by $X/A \cup CA/CA$, which is simply X/A by CA/CA a point. Now note a similar relationship.

$$((X \cup CA) \cup CX)/CX \approx SA$$

In the same fashion, note the intersection of each space in the union with CX and quotient each space by this intersection. The quotient is then rewritten as $X/X \cup CA/A \cup CX/CX$. Note that CA/A is equivalent to SA, for collapsing A to a point incorporates the extra quotient in the definition of the suspension into the space. In other words, this pinches the opposite end of the cone. X/X and CX/CX are both points, so the whole expression reduces to SA. One final relationship follows by the same process.

$$(((X \cup CA) \cup CX) \cup CX) \cup C(X \cup CA))/C(X \cup CA) \approx SX$$

5. Homotopy

An isomorphism between two spaces X and Y is quite rare and often difficult to prove. A more relaxed condition, however, is if the spaces X and Y are *homotopic*, which this section will work towards defining. However, the definition of homotopic functions must come before this definition of homotopic spaces.

Definition 3.30 (Homotopic Functions)**.** Take Topological spaces X and Y with functions $f, g : X \to Y$. Then f *is homotopic to* g if there exists a continuous function $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. The homotopy is denoted by $f \simeq g$.

Homotopy indeed forms an equivalence relation. Informally, two functions are homotopic if one function can be continuously deformed into another function. Consider the following example to build intuition of homotopy.

Example 3.31. Consider the two functions $f : I \to \mathbb{R}^2$ and $g : I \to \mathbb{R}^2$ defined by $f(x) = (x, 0)$ and $g(y) = (0, y)$. Then consider the function $H : I \times [0,1]$ defined by $H(z,t) = ((1-t)x, ty)$. This function is indeed continuous and satisfies $H(x,0) = f(x)$ and $H(y,1) = g(y)$ and thus $f \simeq g$. Visually, f maps the interval onto the x axis whereas g maps the interval onto the y axis. H then carries the image of the interval through the first quadrant from the x axis to the y axis.

The construction of the continuous function H applied a linear transition between the two functions — a common technique.

The homotopy relation is not restricted to functions; two spaces can be homotopic. The idea of relation of homotopic spaces is similar but weaker to an isomorphism of spaces. Recall that two spaces are isomorphic if there exists two continuous functions between the spaces that compose to the identity. Similarly, two spaces are homotopic of two functions compose to a function that is *homotopic* to identity.

Definition 3.32 (Homotopic Spaces)**.** Take topological spaces X and Y. We say X and Y are homotopy equivalent if there exist continuous functions $F : X \to Y$ and $G : X \to Y$ such that $F \circ G \simeq \mathrm{Id}_Y$ and $G \circ F \simeq \mathrm{Id}_X$. Denote the homotopy with $X \simeq Y$.

A homotopy between spaces X and Y is not as strong as an isomorphism, but the two spaces will still share many useful properties. In particular, a homotopy equivalence means that many of the properties learned with K-theory will be the same.

Example 3.33. This example shows $\mathbb{R}^2 \setminus \{(0,0)\} \simeq S^1$. To see this, define $F : \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$ by $F : (x,y) \mapsto (x,y)/|(x,y)|$, which normalizes the point (x,y). Then consider the inclusion mapping $G : S^1 \to \mathbb{R} \setminus \{(0,0)\}$. The composition $F \circ G$ is exactly the identity and thus is homotopic to the identity. For $G \circ F$ consider the linear homotopy $H((x,y),t) = (1-t)(G \circ F)((x,y)) + t\,\mathrm{Id}((x,y))$.

The spaces $\mathbb{R}/\{(0,0)\}$ and S^1 both have some kind of "hole", which is why this homotopy works.

Example 3.34. This example shows $\mathbb{R} \simeq \{0\}$ by considering the function $F : \mathbb{R} \to \{0\}$ by sending everything to 0 and the inclusion function $G : \{0\} \to \mathbb{R}$. Then $F \circ G$ is the identity so it must only be verified that $G \circ F \simeq \mathrm{Id}$ and again the liear homotopy $H(x,t) = (1-t)(G \circ F)(x) + t\,\mathrm{Id}(x)$ works.

Notice that $\mathbb{R}$ is homotopic to a single point. In fact, $\mathbb{R}^n$ is homotopic to a point for any n and so homotopy classes ignore much information about the space. Spaces that are homotopic to a single point often behave trivially and so get a name.

Definition 3.35 (Contractible)**.** A topological space X is contractible if $X \simeq \{pt\}$.

If a space X is contractible, then X shares many of its properties with a point, thus many of the properties of X will be trivial in some sense.

CHAPTER 4

Vector Bundles

To motivate vector bundles, consider any vector field any vector field over $\mathbb{R}^2$ and ask: what larger object is a home for this vector field? A vector field is certainly not a point in $\mathbb{R}^2$, so what larger object does the vector field lie inside of? To identify each vector of a vector field requires four numbers: two numbers (x, y) to identify the location of the vector within the topological space and two additional numbers $\binom{v_x}{v_y}$ to communicate the direction of the vector at this point. This suggests that this vector field rests inside of $\mathbb{R}^2 \times \mathbb{R}^2$ or something similar; denote this $T\mathbb{R}^2$ for now. Interpret $T\mathbb{R}^2$ as the *topological space* $\mathbb{R}^2$ with a copy of the *vector space* $\mathbb{R}^2$ at every point. There is an important distinction between the structure on the two sets $\mathbb{R}^2$. The topological space $\mathbb{R}^2$ is where each point (x, y) resides and the vector space $\mathbb{R}^2$ is where each vector $\binom{v_x}{v_y}$ resides. This distinction opens the door for changing the topological space $\mathbb{R}^2$ to any arbitrary topological space.

Now consider changing the topological space, which is perhaps better called the *base space*, to the sphere S^2. What would a field look like over S^2? Not much changes: a vector field would associate to each point in S^2 some vector in the plane $\mathbb{R}^2$ tangent to the sphere. Then, the whole space that the vector field lives inside is the topological space S^2 with a vector space $\mathbb{R}^2$ associated at every point, which is denoted TS^2.

The examples $T\mathbb{R}^2$ and TS^2 as motivated by vector fields are called *tangent bundles*, which place vector spaces tangent to the space. However, vector bundles in general can be constructed by placing any vector fields on the surface of a base space so long as the placement abides by some rules.

1. Definition and Examples

The definition of vector bundles assigns rules on how to place vector spaces on the surface of a topological object and still get an interesting space.

Definition 4.1 (Vector Bundle)**.** Take X as a topological space. Then, a topological space E paired with a continuous map $p : E \to X$ is a *vector bundle* over X if:

(i) For each $x \in X$, the preimage $p^{-1}(x)$ is a finite vector space with the appropriate subspace topology induced from E.
(ii) E is locally trivial. This requires that for each $x \in X$, there exists an open neighborhood $U \subset X$ containing x such that the preimage of space with respect to the projection mapping is trivial. In other words, $p^{-1}(U) \approx U \times V$ for a vector space V.

The topological space denoted X in the definition is called the *base space* and represents the topological spaces $\mathbb{R}^2$ and S^2 discussed earlier. Then at each point in the base space X, there is the vector space $p^{-1}(x)$ which is called the *fiber* at X and is equivalent to a copy of the vector space $\mathbb{R}^2$ at a point of the sphere discussed previously. A simple example of a vector bundle follows.

Example 4.2 (Cylinder)**.** Take S^1 with the standard topology to be the base space. As a vector space, take $\mathbb{R}$ and consider the vector bundle given by the product $S^1 \times \mathbb{R}$. Giving $\mathbb{R}$ the standard topology induces the product topology on $S^1 \times \mathbb{R}$ and take the projection map $p : S^1 \times \mathbb{R} \to S^1$ given by $p : (x, v) \mapsto x$ to be the continuous projection map. Then, each preimage $p^{-1}(x)$ is a copy of the vector space $\mathbb{R}$ with the appropriate topology and thus this gives a vector bundle. In fact, this vector bundle should be visualized as a cylinder.

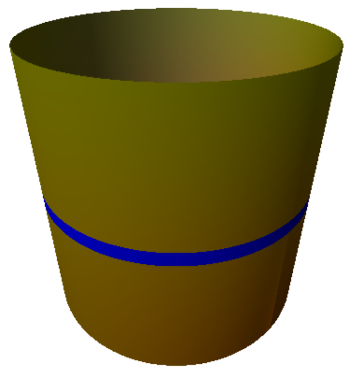

FIGURE 1. Cylinder as a Vector Bundle

The above construction of the cylinder demonstrates that fibers need not be tangent to the base space, but in fact fibers are not required to correspond to the dimension of the base space. The example of the cylinder is a specific case of the idea of a *trivial bundle* which is defined as follows.

Definition 4.3 (Trivial Bundle)**.** Let X be a topological base space and let V be a vector space with a topology. Then, taking the product topology, $X \times V$ forms a topological space. This together with the projection map $p : X \times V \to X$ given by $p : (x, v) \mapsto x$ forms a vector bundle. This vector bundle is called a *trivial bundle*. If E is of dimension n, the trivial bundle is often denotes ε^n.

With this construction of the trivial bundle, note the "$X \times V$" within the "locally trivial" condition of Definition 4.1 is understood to have the structure of the trivial bundle. Trivial bundles, such as the cylinder, automatically satisfy this condition. In general, however, this local triviality condition must be verified by constructing an isomorphism to a trivial bundle for each local region. However, in order to construct such an isomorphism, we must first complete the category of vector bundles by providing a notion of morphisms between vector bundles. Vector bundles contain the structure of both a topological space and of many vector spaces, so a homomorphism of vector bundles aims to preserve both of these structures. These homomorphisms will be between bundles with the same base space and are defined as follows.

Definition 4.4 (Homomorphisms of Vector Bundles)**.** Take two vector bundles E and F both with base space over X. Then, let $p : E \to X$ and $q : F \to X$ be the continuous maps. A mapping $\varphi : E \to F$ is a *homomorphism of vector bundles* if:

(i) $q\varphi = p$

(ii) $\varphi : E \to F$ is a homomorphism of topological spaces; that is, φ is continuous.
(iii) For each $x \in X$, the mapping $\varphi : p^{-1}(x) \to q^{-1}(x)$ is a homomorphism of vector spaces; that is, φ is a linear map between these vector spaces.

The definition of isomorphism for vector bundles carries over from the definition of isomorphism in category theory: a homomorphism with a homomorphism as an inverse. However, some of the homomorphism properties of the inverse follow automatically. For instance, take vector bundles $p : E \to X$ and $q : F \to X$ and a bijective homomorphism $\varphi : E \to F$. It follows immediately from $q\varphi = p$ that $p\varphi^{-1} = q$, so this does not need to be checked. Additionally, a bijective linear map will have a linear inverse. Then, it does not need to be verified that φ^{-1} maps the fibers in a linear way because it is known that φ does. However, the continuous property of φ^{-1} does not follow automatically and is typically the most difficult part of isomorphism proofs. With these observations, an isomorphism can be defined in the following more practical way.

Definition 4.5 (Isomorphism)**.** For two vector bundles $p : E \to X$ and $q : F \to X$, a map $\varphi : E \to F$ is defined to be an isomorphism if it is a bijective homomorphism with continuous inverse.

Typically, the most difficult part of verifying a map is an isomorphism is the continuous inverse condition. Due to this, it is most convenient to verify isomorphisms with the following result.

Lemma 4.6. For two vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ over the same base space, a continuous map $h : E_1 \to E_2$ is an isomorphism if it takes each fiber $p_1^{-1}(x)$ to the corresponding fiber $p_2^{-1}(x)$ by an isomorphism of vector spaces.

According to this Lemma, having that each fiber is an isomorphism is enough to know that a continuous map is an isomorphism. This again hints that a vector bundle is simply a collection of vector spaces with some extra topological structure. [**4**, p. 8] gives a technical proof of this result.

The most common isomorphism to see is a *trivialization* isomorphism — an isomorphism to the trivial bundle, which is necessary to verify the local triviality condition in the following examples. The example of a Möbius band as a vector bundle is the simplest example of a nontrivial vector bundle.

Example 4.7 (Möbius Strip)**.** Consider the vector bundle $M = I \times R/\sim$ where the equivalence relation $\sim$ relates every point to itself with the additional relation $(0, v) \sim (1, -v)$. This definition provides an easy choice of projection map $p : M \to S^1$ through the map $p : (x, v) \mapsto x$. Note that $I/\sim$ is isomorphic to S^1 and so this bundle has the base space of S^1 with one dimensional vector spaces. However, the difference is the subtle "$-$" in the equivalence relation, which gives a twist to the topological space. In fact, the bundle M gives a Möbius band.

It still must be explicitly shown that M is indeed a vector bundle. First, note that for each $x \in S^1$, the construction gives $p^{-1}(x) \cong \mathbb{R}$ and $\mathbb{R}$ is indeed a vector space. For the local triviality condition, fix any point $x \in S^1$. So long as $x \neq 0$, there is an open set $U \subset I$ containing x. In this case, $p^{-1}(x) = U \times \mathbb{R}$ by definition giving immediate local triviality. In the case that $x = 0$, a specific *local trivialization* must be constructed. That is, an isomorphism to the trivial bundle. In this case, consider the open set $U = (1/2, 1) \cup [0, 1/2)$ together with the local trivialization $\varphi : p^{-1}(U) \to U \times \mathbb{R}$ defined by $\varphi : (x, v) \mapsto (x, v)$ if $\varphi \in [0, 1/2)$ and $p : (x, v) \mapsto (x, -v)$ otherwise. This is an indeed continuous by the definition of the equivalence relation and this map acts as its own inverse, which gives φ is an isomorphism. A visual for intuitively understanding that a Möbius band is locally trivial is given in Figure 2. In the figure, the point $x \in S^1$ is represented by a green point, the open set $U \subset S^1$ containing x is magenta and the region $p^{-1}(U)$ is red.

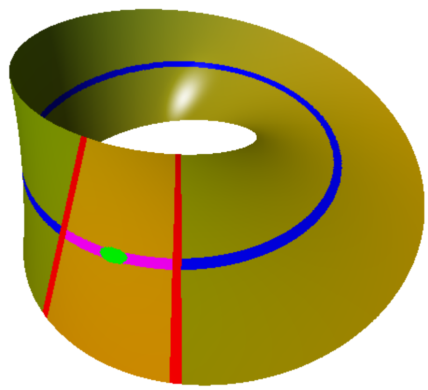

FIGURE 2. Möbius Band is Locally Trivial

The dimension of a vector bundle refers to the dimension of the vector space placed over the base space, so both the cylinder and the Möbius band are of dimension 1. Dimension 1 bundles are referred to as *line bundles*. Note that so long as the base space is connected, the local triviality condition forces all of the vector spaces in the bundle to be equivalent, ensuring the dimension is well-defined. The following example constructs a bundle of dimension 2.

Example 4.8 (TS^2)**.** The *tangent bundle* to S^2 is denoted TS^2 and is defined as the set $E = \{(x, v) \in S^2 \times \mathbb{R}^3 : \langle x, v\rangle = 0\}$ where the inner product is defined by using $S^2 \subset \mathbb{R}^3$ to consider elements of the sphere as vectors. This goes together with the natural projection map $p : E \to S^2$ defined by $p : (x, v) \mapsto x$.

To show that TS^2 is indeed a vector bundle, note that the condition $\langle x, v\rangle = 0$ ensures that $p^{-1}(x)$ will form a vector subspace for each $x \in X$. For the local triviality condition, fix any point $x \in S^2$ and consider the open hemisphere $U \subset S^1$ centered at x. Now define the local trivialization $\varphi : p^{-1}(U) \to U \times \mathbb{R}^2$ by $\varphi : (y, v) \mapsto (y, \text{proj}_{p^{-1}(x)}(v))$ where $\text{proj}_{p^{-1}(x)}(v)$ denotes the orthogonal projection of v onto the vector space $p^{-1}(x)$. This is a continuous map with isomorphisms between the corresponding fibers, thus Lemma 4.6 promises that this is indeed an isomorphism to the trivial bundle.

Example 4.9 (NS^2)**.** The *normal bundle* to S^2 is denoted NS^2 and is defined as the set $E = \{(x, v) \in S^2 \times \mathbb{R}^3 : v \in \text{Span}(x)\}$ where again x can be considered as a vector by $S^2 \subset \mathbb{R}^3$. The bundle E comes equipped with the projection map $p : E \to S^2$ such that $p : (x, v) \mapsto x$.

The verification for local triviality uses a similar construction as the tangent bundle. Again consider an open hemisphere U for each $x \in S^2$ and define $\varphi : p^{-1}(U) \to U \times \mathbb{R}$ by $\varphi : (y, v) \mapsto (y, \text{proj}_{p^{-1}(x)}(v))$, and Lemma 4.6 gives that this is an isomorphism.

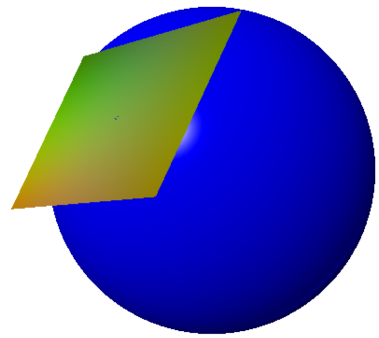

FIGURE 3. Tangent Vector Space at one point of S^2

Recall that the real projective line considers equivalence classes in $\mathbb{R}^2 \setminus \{(0,0)\}$ where each class represents a line through the origin that does not contain the origin. Taking the line that each point in $\mathbb{R}P^1$ represents to be the fiber over that point is a natural construction of a vector bundle. This construction extends to any projective space $\mathbb{R}P^n$ and the resulting bundle is called the *canonical line bundle* over the space. The case of $\mathbb{R}P^1$ is given in more detail below.

Example 4.10 (Canonical Line Bundle over $\mathbb{R}P^1$)**.** Let $\mathbb{R}P^1$ be the real projective line and define a vector bundle $p : E \to \mathbb{R}P^1$ by

$$E = \left\{([x], v) \in \mathbb{R}P^1 \times \mathbb{R}^2 : v \in \text{Span}(x)\right\}$$

together with the natural projection map $p : ([x], v) \mapsto [x]$. The local trivialization is again given by an orthogonal projection.

Vector bundles need not be over a real vector space $\mathbb{R}^n$. In fact, taking the vector space to be a complex vector space $\mathbb{C}^n$ provides some pleasant patterns. Note the following example of a bundle that takes the vector space $\mathbb{C}$.

Example 4.11 (Canonical Line Bundle over $\mathbb{C}P^1$)**.** The canonical line bundle over $\mathbb{C}P^1$ is the bundle $p : H \to \mathbb{C}P^1$ such that

$$H = \left\{([x], v) \in \mathbb{C}P^1 \times \mathbb{C}^2 : v \in \text{Span}(x)\right\}$$

together with the natural projection map. The local trivialization is constructed by an orthogonal projection.

The canonical line bundle plays a central role in K-theory; it is the isomorphism $\mathbb{C}P^1 \approx S^2$ together with this particular that will give K-theory pleasant patterns on spheres.

2. Direct Sum and Tensor Product on Vector Bundles

Again take the view of a vector bundle as a collection of vector spaces with some extra topological properties. In fact, every point of a vector bundle E belongs to some fiber of the bundle, so the E as a set is simply the disjoint union of only fibers. By taking this perspective, much of the structure of vector spaces extends to vector bundles. For instance, the fibers can be used to construct the direct sum operation in the following way.

Definition 4.12 (Direct Sum of Vector Bundles)**.** Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles over X. Then, consider the disjoint unions of the direct sums of fibers

$$E_1 \oplus E_2 = \bigcup_{x \in X} p_1^{-1}(x) \oplus p_2^{-1}(x)$$

together with the projection mapping $p : E_1 \oplus E_2 \to X$ given by $p : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto x$. Then, $p : E_1 \oplus E_2 \to X$ when given a natural topology forms a vector bundle over X called the *direct sum* of E_1 and E_2.

Of course, a vector bundle has more structure than simply a union of vector spaces; in particular, vector bundles must be given a topology and must satisfy the local triviality condition. The verifications at the end of this chapter provide the specific details of this "natural topology" referred to in the above definition along with this necessary proof of local triviality, but these verifications all work out. Because a vector bundle is built out of fibers, vector space properties such as the direct sum carry over naturally to vector bundles and the extra properties typically "all work out".

In this construction, consider some $x \in X$ and let $v_1 \in p_1^{-1}(x)$ and $v_2 \in p_2^{-1}(x)$ be elements of both fibers. Then, taking the direct sum of these vector spaces, these two vectors can be identified with $v_1 \oplus v_2$, which can also be thought of as simply (v_1, v_2). Elements of the direct sum bundle can be thought of as an ordered pair containing of elements of each original bundle.

A second similar construction by using the fibers is in the extension of the tensor product to vector bundles.

Definition 4.13 (Tensor Product of Vector Bundles)**.** Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles over X. Then, consider the disjoint unions of all tensor products of the fibers

$$E_1 \otimes E_2 = \bigcup_{x \in X} p_1^{-1}(x) \otimes p_2^{-1}(x)$$

together with the projection mapping $p : E_1 \otimes E_2 \to X$ given by $p : p_1^{-1}(x) \otimes p_2^{-1}(x) \mapsto x$. Then, $p : E_1 \otimes E_2 \to X$ when given a natural topology forms a vector bundle over X called the *tensor product* of E_1 and E_2.

Again, the specifics of the "natural topology" and the verification of natural triviality all work out as explained in the verifications at the end of the chapter. The proof is identical to the proof for direct sum. It is worth mentioning that this construction can be generalized to other operations on vector spaces such as the dual and the exterior power, but these notes only require the direct sum and the tensor product.

Because the tensor product and direct sum are defined on each fibers, the properties of direct sum and tensor product on vector spaces carry over to analogous properties on vector bundles.

Claim 4.14. Listed below are properties of direct sum and tensor product over vector bundles.

(i) The direct sum between bundles is associative and commutative.
(ii) The trivial bundle of dimension 0 is an identity element for the direct sum. That is, $E \oplus \varepsilon^0 = E$.
(iii) The tensor product between bundles is associative and commutative.

(iv) The trivial dimension of dimension 1 is an identity element for the tensor product. That is, $E \otimes \varepsilon^1 = E$.
(v) The tensor product distributes over direct sum.

The proof verifying each of these properties must show an isomorphism between bundles. The full proof is given at the end of the chapter, but the difficult part of is to show the claimed isomorphism satisfies the continuity conditions. Luckily, however, continuity is a *local property*, meaning that it suffices to verify continuity for an open cover of small neighborhoods. This strategy works well with vector bundles because the local triviality condition promises that so long as the neighborhoods are small enough, each neighborhood will be trivial. After this key observation, the rest of the proof follows.

3. Pullback Bundles

The following construction addresses pullback bundles. In the next chapter of this story, all of the arrows will suddenly point backwards as a contravariant functor emerges. The reason why the arrows will point backwards is due to pullback bundles.

Consider two base spaces X and Y where X has a vector bundle structure $p : E \to X$ but Y, unfortunately, has no such structure. However, Y can be given a vector bundle $q : F \to Y$ by stealing the structure of E through the association given by f. Specifically, assign the fiber over each point $y \in Y$ as an exact copy of the fiber over $f(y)$.

Definition 4.15 (Pullback Bundle)**.** Let $f : X \to Y$ be a mapping and $p : E \to X$ a bundle as defined above. Then there exists a unique bundle $f^*(p) : f^*(E) \to Y$ and a mapping $h : f^*(E) \to E$ such that h maps each fiber $(f^*(p))^{-1}(y)$ to the fiber $p^{-1}(f(y))$ as a vector space isomorphism. This bundle is called the *pullback bundle* and denoted $f^*(p) : f^*(E) \to Y$.

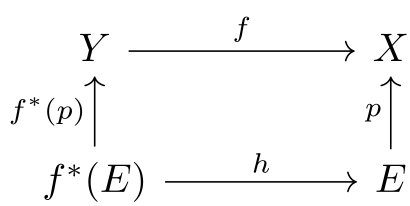

FIGURE 4. Pullback Bundle Commutative Diagram

Again existence and uniqueness must be verified. Following [**4**, p. 18-19], the existence proof considers the construction

$$F = \{(y, e) \in Y \times E : f(y) = p(e)\}$$

which is worth particular attention, for many sources such as [**1**] use this construction as the definition. It must be verified that F is both a vector bundle and that F satisfies the defining property of the pullback. This requires defining a projection map $q : F \to Y$ by $q : (y, e) \mapsto e$. The vector space structure on each $q^{-1}(y)$ is defined by the vector space structure on $p^{-1}(f(y))$ by

$$\alpha(y, v) + \beta(y, w) = \alpha(y, v + w)$$

These definitions indeed make F into a valid vector bundle, and F indeed satisfies the defining property of pullback by considering the mapping $h : F \to E$ defined by $h : (y, e) \mapsto e$. Note that the condition $f(y) = p(e)$ in F is constructed exactly so that Figure 4 commutes.

$$(p \circ h)((y, e)) = p(e) = f(y) = (f \circ q)((y, e))$$

For uniqueness, assume that there is some bundle $q' : F' \to Y$ with a map $h' : F' \to E$ that satisfies the universal property of pullback. But now consider the specific pullback F constructed in an existence proof and the mapping $\varphi : F' \to F$ by $\varphi : f \mapsto (q'(f), h'(f))$. By the properties of h', this mapping will act as a vector space isomorphism on each fibers, so Lemma 4.6 gives that $F' \approx F$, giving uniqueness.

However, there is a detail of well-defined to address. When given a vector bundle $p : E \to X$ with continuous functions $f : X \to Y$ and $g : Y \to Z$, how is the bundle structure on X pulled back to a bundle on Z? There are two options: $(f \circ g)^*(E)$ and $f^*(g^*(E))$. Luckily, the following claim shows that the two options are isomorphic and gives more pleasant properties of the pullback.

Claim 4.16. Listed below are important properties of pullbacks.

(i) $(f \circ g)^*(E) \approx g^*(f^*(E))$ for any bundle E and continuous functions f and g.
(ii) $\mathrm{Id}^*(E) \approx E$ for any vector bundle E over X and the identity mapping $\mathrm{Id} : X \to X$.
(iii) $f^*(\varepsilon^n) \approx \varepsilon^n$ for all continuous functions f and trivial bundles ε^n over the corresponding base spaces.
(iv) $f^*(E_1 \oplus E_2) \approx f^*(E_1) \oplus f^*(E_2)$ for all bundles E_1 and E_2 and continuous function f.
(v) $f^*(E_1 \otimes E_2) \approx f^*(E_1) \otimes f^*(E_2)$ with E_1 and E_2 vector bundles and f a continuous function.

A full verification for each of these properties is given at the end of the chapter, but the strategy is to show isomorphism by demonstrating that the defining property of pullback is satisfied and then appealing to uniqueness. Recall that a pullback bundle $f^*(E)$ assigns a fiber to each point y in its base space by stealing the structure of the fiber over $f(y)$. However, if f collapsed the domain to a single point, every point y will be assigned the same fiber, resulting in a trivial bundle as in the following example.

Example 4.17. Take any bundle $p : E \to X$ together with the continuous function $f : Y \to X$ between topological spaces that maps all of Y to a single point x_0; that is, $f : y \mapsto x_0$. Note that the pullback defines each fiber over a point $y \in Y$ to steal the structure of $f(y)$, so in this case the pullback $f^*(E)$ is given by the trivial bundle $Y \times p^{-1}(x_0)$. To formally verify this is the trivial bundle, consider the mapping $h : f^*(E) \to E$ given by $h : (y, v) \mapsto (x_0, v)$. This takes each fiber over a point $y \in Y$ to the fiber over x_0 by the identity mapping, for they are defined to be the same vector space. The identity mapping is indeed a vector space isomorphism over the fibers, so h fulfills the defining property of pullback.

Example 4.18 (Restriction)**.** Take any vector bundle $p : E \to X$ and a subspace $A \subset X$, which comes with the inclusion map $i : A \to X$. Then, the pullback $i^*(E)$ is given by the *restriction* of E to A, which is simply the space notated by $p^{-1}(A)$ previously. To verify this, consider the mapping $h : p^{-1}(A) \to E$ defined by an inclusion $h : e \mapsto e$, for $p^{-1}(A)$ is a subspace of E. This identity mapping indeed acts as a vector space isomorphism over the fibers, so the restriction $p^{-1}(A)$ fulfills the defining property of the pullback.

Note that given a vector bundle $p : E \to X$, a function f from Y *to* X is necessary to induce a pullback bundle $f^*(E)$ over Y; not the other way around. The function must point this direction in order for $f^*(E)$ to effectively steal the structure of E. A function $f' : X \to Y$ would be rather useless, for this function may associate each point of the base space Y to multiple points in the base space X or non at all. However, given the function $f : Y \to X$, each point in Y is mapped to a single point in X and so the structure of E can be effectively stolen. This fact — that a function induces a vector bundle in the opposite direction — is half way to defining a contravariant functor.

4. Necessary Results on Vector Bundles

Note that the properties of direct sum and tensor product on vector bundles does not include a notion of additive inverse. The following result is crucial to eventually getting the desired additive inverse property.

Claim 4.19. For every bundle E over a compact Hausdorff space X, there exists a bundle E' over X such that $E \oplus E'$ is trivial.

A (lengthy) full proof of the claim is given in [**4**], but the idea is as follows. Given the bundle $p : E \to X$ over a compact Hausdorff space, a huge trivial bundle T is constructed by using a topology theorem[1] that follows from the compact Hausdorff condition. The trivial bundle T is built exactly such that there is a convenient isomorphism from E to a sub-bundle E_0 in the huge trivial bundle. Another topology tool[2] allows the extension of an inner product to vector bundles, which then gives a Gran-Schmidt orthogonalization process on vector bundles. The orthogonal complement of each fiber in E_0 gives a vector bundle $E_0^\perp$ such that $E_0 \oplus E_0^\perp = T$ and the desired conclusion follows from $E \cong E_0$.

Example 4.20. For an example of the above theorem, consider the tangent bundle to S^2, denoted TS^2, which satisfies $TS^2 \oplus NS^2$ trivial. To see this, consider the space S^2 as embedded inside $\mathbb{R}^3$. Then elements of TS^n can be expressed $(x, v) \in S^2 \times \mathbb{R}^3$ and similarly, elements of NS^2 are given by $(x, n) \in S^2 \times \mathbb{R}^3$. Further, at a fixed point x, all vectors v in the tangent fiber will be orthogonal to the vectors n in the normal fiber by the definition of the bundles. Then elements of the direct sum $TS^2 \oplus NS^2$ can be expressed by $(x, v \oplus n)$ or simply (x, v, n). Then consider the isomorphism $\varphi : TS^2 \oplus NS^2 \to S^2 \times \mathbb{R}^3$ given by the isomorphism.

$$\varphi : (x, v, n) \mapsto (x, v + n)$$

The above mapping an isomorphism follows from the above continuous and a linear bijection. The inverse map to the above can be constructed by taking the projection of the vector component onto the normal and tangent subspaces, which is again continuous giving isomorphism.

Homotopy relationships are closely related to the structure of a vector bundle. Consider the following result, for example.

Claim 4.21. Take a vector bundle $p : E \to X$. If the base space X is contractible, then the bundle E is trivial.

---

[1] Urysohn's Lemma
[2] Partition of Unity

Any vector bundle over a point must be trivial, and a contratibe space many properties with a point. A full proof of this result is given in [**1**, p. 18].

Claim 4.22. If H is the canonical line bundle over $\mathbb{C}P^1$, then $(H \otimes H) \oplus 1 \approx H \oplus H$.

The justification for this theorem, following [**4**, p. 24], begins with $\mathbb{C}P^1 \approx S^2$ as discussed in Example 3.13, revealing that the canonical line bundle H can be seen as a bundle over S^2. The verification for Claim 4.22 uses the tool of *clutching functions*, which constructs vector bundles on spheres The idea of clutching function is that the sphere S^2 is made from joining two disks D_2^+ and D_2^-, corresponding to two hemispheres. Next, consider the restriction of the bundle H over the disks D_2^+ and D_2^- and note that disks are contractible and thus the restrictions will reduce to the trivial bundle by Claim 4.21. The bundle H can then be rebuilt by considering the two trivial bundles $D_2^+ \times \mathbb{C}$ and $D_2^- \times \mathbb{C}$ together with a description of how the two halves fit together. This description is formally given by a mapping from $f : S^1 \to GL_1(\mathbb{C})$, where $S^1 \subset S^2$ is where the two hemispheres meet. The bundle H is then reconstructed by identifying (x, v) with $(x, f(x)v)$ for some *clutching function* f. In the case of the canonical line bundle, $f = (z)$. An important property of clutching functions is that if bundle E_f is described by clutching function f and bundle E_g is described by clutching function g and f is homotopic to g, then $E_f \approx E_g$.

Now recall the definition of direct sum on vector bundles to construct the clutching function f on $H \oplus H$ and g on $(H \otimes H) \oplus 1$ where $f, g : S^1 \to GL_2(\mathbb{C})$. In particular, f must satisfy $f(z)(v \oplus v) = (z)(v) \oplus (z)(v)$ and g must satisfy $g(z)((v \otimes v) \oplus 1) = (z)(v) \otimes (z)(v) \oplus 1$. These clutching functions must then be as follows.

$$f : z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \quad \text{and} \quad g : z \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$$

However, note that $f(z)$ and $g(z)$ only differ by scaling. In other words, there is a homotopy $H : S^1 \times I \to GL_2(\mathbb{C})$, which is explicitly given by $H : (z, x) \mapsto z^{-x} f(z)$. The homotopy $f \simeq g$ then gives $H \oplus H \approx (H \otimes H) \oplus 1$.

5. Verifications

5.1. Direct Sum and Tensor Product Verifications.

It must be verified that the direct sum has a natural topology that indeed makes it a vector bundle.

Proof. Take vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ and recall that the direct sum on bundles as a set is given by the disjoint union of direct sums on fibers

$$E_1 \oplus E_2 = \bigcup_{x \in X} p_1^{-1}(x) \oplus p_2^{-1}(x).$$

This set is paired with with the projection $p : E_1 \oplus E_2 \to X$ given by $p : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto x$.

The topology on $E_1 \oplus E_2$ is defined in this paragraph. For each $x \in X$, the definition of vector bundle promises an open set U containing x over which both E_1 and E_2 are trivial. This provides trivializations $t_1 : p_1^{-1}(U) \to U \times V_1$ and $t_2 : p_2^{-1}(U) \to U \times V_2$ for vector spaces V_1 and V_2. Next, define the map $t_1 \oplus t_2 : p_1^{-1}(U) \oplus p_2^{-1}(U) \to U \times (V_1 \oplus V_2)$ as follows.

$$t_1 \oplus t_2 : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto t_1(p_1^{-1}(x)) \oplus t_2(p_2^{-1}(x))$$

Then, the topology on $p_1^{-1}(U) \oplus p_2^{-1}(U)$ is defined by requiring the map $t_1 \oplus t_2$ to be a homeomorphism. By letting x vary, this defines a topology over all of $E_1 \oplus E_2$. It must be verified, however, that this topology is well-defined.

Before the proof of well-defined, observe how this choice of topology gives that $E_1 \oplus E_2$ is a vector bundle. Firstly, this choice equips each fiber $p_1^{-1}(x) \oplus p_2^{-1}(x)$ with the typical topology of the direct sum of vector spaces. This ensures that the projection map $p : E_1 \oplus E_2 \to X$ is continuous. Next, the local triviality condition must be verified. Luckily the topology is built exactly so that $t_1 \oplus t_2$ is a trivialization. For any $x \in X$, the mapping $t_1 \oplus t_2$ defined on the appropriate U as described above satisfies all the conditions of a vector bundle homomorphism. Further, the defining condition that $t_1 \oplus t_2$ is a homeomorphism promises a continuous inverse and so $t_1 \oplus t_2$ is an isomorphism of vector bundles.

It only remains to show that the topology on $E_1 \oplus E_2$ is well-defined. In particular, it must be shown that the topology is independent of the choice of trivializations over a single open set U and that the open sets induce the same topology over their intersection. So, for $x \in X$ and corresponding $U \subset X$, consider two trivializations for each bundle: $t_1, t_1' : E_1 \mapsto U$ and $t_2, t_2' : E_1 \mapsto U$. Because each trivialization gives an isomorphism to the trivial bundle, the composition $t_1^{-1} \circ t_1' : p^{-1}(U) \to p^{-1}(U)$ is an isomorphism and similarly $t_2^{-1} \circ t_2' : p^{-1}(U) \to p^{-1}(U)$ is an isomorphism. Then composition $t_1' \circ t_1^{-1}$ is an isomorphism on $U \times V_1$ and similarly $t_2' \circ t_2^{-1}$ is an isomorphism on $U \times V_2$. It follows that the composition $(t_1' \oplus t_2') \circ (t_1 \oplus t_2)^{-1}$ is an isomorphism on $U \times (V_1 \otimes V_2)$, which implies that the choices $(t_1 \oplus t_2)$ and $(t_1' \oplus t_2')$ supply the same topology.

Finally, consider a separate set of open set $U' \subset X$. Then, taking the restrictions of the bundles $p^{-1}(U)$ and $p^{-1}(U')$ over the intersection $U \cap U'$ would only differ in the trivializations, which induce the same topology as shown in the previous paragraph. □

In the above argument, the only part that appeals to the direct sum operation itself is the implicit assumption that the mapping $(v, w) \mapsto v \oplus w$ is continuous. This is also true for the tensor product, so a simple substitution of "$\otimes$" in place of "$\oplus$" in the above proof provides the needed verification for tensor product.

Proof of Claim 4.14. Verifying each claim requires establishing an isomorphism φ over two bundles, say $p : E \to X$ and $q : F \to X$. The approach will be to establish a vector space isomorphism between the fibers, which gives necessary properties of vector bundle isomorphism except for continuity and continuity of inverse. To deal with the continuity conditions, note that continuity is a local condition. Thus it suffices to show that for every $x \in X$, there is an open neighborhood U such that the restricted function $\varphi : p^{-1}(U) \to q^{-1}(U)$ is continuous in both directions.

(i) For associativity of the direct product, consider vector bundles E_1, E_2, E_3 over a base space X with corresponding projection maps p_1, p_2, and p_3. An isomorphism $\varphi : (E_1 \oplus E_2) \oplus E_3 \to E_1 \oplus (E_2 \oplus E_3)$ must be constructed. Let φ be the linear bijective function defined on the fibers by

$$\varphi : (p_1^{-1}(x) \oplus p_2^{-1}(x)) \oplus p_3^{-1}(x) \mapsto p_1^{-1}(x) \oplus (p_2^{-1}(x) \oplus p_3^{-1}(x))$$

For the continuity conditions, fix a point $x \in X$. Then, choose an open set $U \subset X$ small enough such that the local triviality conditions are satisfied by both direct sum bundles. Then, noting the vector space isomorphism $(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3)$, continuity in both directions is given by the following composition of isomorphisms

$$(p_1^{-1}(U) \oplus p_2^{-1}(U)) \oplus p_3^{-1}(U) \to U \times (V_1 \oplus V_2) \oplus V_3$$
$$\to U \times V_1 \oplus (V_2 \oplus V_3) \to p_1^{-1}(U) \oplus (p_2^{-1}(U) \oplus p_3^{-1}(U))$$

The proof for commutativity follows in a near identical way. The difference being that an isomorphism $\varphi : E_1 \oplus E_2 \to E_2 \oplus E_1$ is considered with the mapping between fibers $\varphi : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto p_2^{-1}(x) \oplus p_1^{-1}(x)$ and the vector space isomorphism $V_1 \oplus V_2 \cong V_2 \oplus V_1$ is considered instead.

(ii) Verifying that ε^0 is the identity element under direct sum requires establishing an isomorphism $\varphi : E \oplus \varepsilon^0 \to E$. This follows in the same way as the previous claims, but uses the mapping of fibers $\varphi : p^{-1}(x) \oplus \{0\} \mapsto p^{-1}(x)$ and uses the vector space isomorphism $V \oplus \{0\} \cong V$.

(iii) The proofs for associativity and commutativity of the tensor product is given by a substitution of "$\otimes$" for "$\oplus$" in the corresponding direct sum proofs.

(iv) The proof that ε^1 acts as an identity element over the tensor product follows similarly to the identity proof over direct sum. The difference being that here an isomorphism $\varphi : E \otimes \varepsilon^1 \to E$ is established by the mapping of fibers $\varphi : p^{-1}(x) \otimes V^1 \mapsto p^{-1}(x)$ where V^1 represents a one dimensional vector space. This proof additionally uses the vector space isomorphism $V \oplus V^1 \cong V$.

(v) Finally, the proof for distributivity establishes a vector space isomorphism $\varphi : E_1 \otimes (E_2 \oplus E_3) \to (E_1 \otimes E_2) \oplus (E_1 \otimes E_3)$ given by the linear bijection on the fibers

$$\varphi : p_1^{-1}(x) \otimes (p_2^{-1}(x)) \oplus p_3^{-1}(x)) \mapsto p_1^{-1}(x) \otimes p_2^{-1}(x) \oplus p_1^{-1}(x) \otimes p_3^{-1}(x)$$

and later uses the isomorphism on vector spaces $V_1 \otimes (V_2 \oplus V_3) \cong (V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$.

□

5.2. Pullback Bundle Verifications.

Proof of Claim 4.16. The strategy for proving each of the following isomorphisms is to take advantage of the uniqueness property. If it can be shown that one side of the isomorphism satisfies the defining property of pullback for the other side, then they must be isomorphic by uniqueness.

(i) For topological spaces X, Y, Z let $g : Z \to Y$ and $f : Y \to X$ be continuous functions and let $p : E \to X$ be a vector bundle. By definition, the bundles $f^*(E)$ and $g^*(f^*(E))$ come equipped with maps $h_g : g^*(f^*(E)) \to f^*(E)$ and $h_f : f^*(E) \to E$ that isomorphically map fibers to corresponding fibers. Then, the composition $h_f \circ h_g : g^*(f^*(E)) \to E$ isomorphically maps fibers to corresponding fibers. Further, the bundle $g^*(f^*(E))$ comes equipped with a projection mapping r into the base space Z. Thus, the triple $g^*(f^*(E))$, $h_f \circ h_g$, and r satisfy the defining characteristics of the pullback bundle $(f \circ g)^*(E)$, giving isomorphism by uniqueness.

(ii) Take the mapping $\mathrm{Id} : X \to X$ for a topological space X with a bundle $p : E \to X$. Then, the bundle E itself with the identity mapping $\mathrm{Id} : E \to E$ isomorphically maps fibers to fibers and comes equipped with the projection mapping p to X. Then, the triple E, $\mathrm{Id} : E \to E$, and p satisfy the defining characteristics of the pullback $\mathrm{Id}^*(E)$ which promises the isomorphism $E \cong \mathrm{Id}^*(E)$ by uniqueness.

(iii) Let $f : Y \to X$ be a continuous function between topological spaces and consider the trivial bundle $p : X \times V \to X$ over X with the regular projection p. Then, consider the trivial bundle over $q : Y \times V \to Y$ over Y with the regular projection q. Then, the mapping $h : Y \times V \to X \times V$ given by $h : (y, v) \mapsto (f(y), v)$ gives the identity mapping over each fiber and is thus a linear isomorphism of the fibers. Thus, the triple $Y \times V$, h, and q satisfies the defining properties of $f^*(X \times V)$ and thus uniqueness promises an isomorphisms between the trivial bundles $Y \times V \cong f^*(X \times V)$. Note that the trivial pullback is over the same vector space.

(iv) Next, take $f : Y \to X$ to be a continuous function between topological spaces. Further, let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles. The pullbacks $f^*(E_1)$ and $f^*(E_2)$ then come with mappings $h_1 : f^*(E_1) \to E_1$ and $h_2 : f^*(E_2) \to E_2$ that are isomorphisms on the fibers. Then, the direct sum of the pullbacks has a mapping $h : f^*(E_1) \oplus f^*(E_2) \to E_1 \oplus E_2$ defined on the fibers by $h : p_1^{-1}(x) \oplus p_2^{-1}(x) \mapsto h_1(p_1^{-1}(x)) \oplus h_2(p_2^{-1}(x))$ which is also an isomorphism on the fibers. Additionally note that the direct sum comes equipped with a projection mapping p onto Y. Thus the triple $f^*(E_1) \oplus f^*(E_2)$, h, and p satisfy the defining properties of the pullback $f^*(E_1 \oplus E_2)$ giving the isomorphism $f^*(E_1) \oplus f^*(E_2) \cong f^*(E_1 \oplus E_2)$ by uniqueness.

(v) The proof for the distributivity of pullback over tensor product is identical to preceding such proof for direct sum, differing only by replacing each "$\oplus$" with "$\otimes$".

□

CHAPTER 5

The Definition of K-Theory

K-Theory is a functor from the category of topological spaces to the category of rings. Topological spaces are messy, making it difficult to understand properties about topological spaces and homomorphisms between topological spaces. However, groups and rings are simple algebraic objects with much structure — an easier object to analyze.

The K-theory functor first considers all possible vector bundles over a topological space. Looking at every possible bundle is too much information, but it turns out that after simplifying the set to equivalence classes of vector bundles, a ring structure emerges.

There are two veins of K-theory; the difference is in the equivalence classes used to reduce the set of vector bundles. First, there is the *K-theory* of a topological space X, denoted $K(X)$. In this case, the equivalence classes have a natural semiring structure and the ring is defined through the ring extension. Secondly, there is the *reduced K-theory* of a topological space X, denoted $\widetilde{K}(X)$, which has bigger equivalence classes. In reduced K-theory, the equivalence classes themselves can be made directly into a ring. In both K-theory and reduced K-theory, the functor is contravariant.

1. The K-Theory Functor K

K-theory aims to construct a ring on the set of vector bundles by making use of the direct sum $\oplus$ and the tensor product $\otimes$ operations. However, this will not give a notion of additive inverses, providing only a semiring. As seen in Chapter 2, a commutative semiring has a unique extension if there is an additive cancellation property. Such an additive cancellation property would means that $E_1 \oplus F \approx E_2 \oplus F$ would imply $E_1 \approx E_2$, which is not necessarily the case. However, applying Claim 4.19 brings us closer to an additive cancellation property by taking the promised bundle F' such that $F \oplus F'$ is trivial as seen in the following computation.

$$E_1 \oplus F \approx E_2 \oplus F \implies E_1 \oplus (F \oplus F') \approx E_2 \oplus (F \oplus F') \implies E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^n$$

This computation motivates the following equivalence relation, which will place E_1 and E_2 inside the same equivalence class, ultimately giving the additive cancellation property.

Definition 5.1 (Stably Isomorphic)**.** Define the equivalence relation $\approx_s$ on vector bundles over the same base space such that for bundles E_1 and E_2, $E_1 \approx_s E_2$ if $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^n$ for some n where ε^n denotes the n dimensional trivial bundle. Here, E_1 and E_2 are said to be *stably isomorphic.*

This equivalence relation gives a natural semiring structure on the equivalence classes.

Claim 5.2. Take compact Hausdorff base space X. The set of all stably isomorphic equivalence classes over the vector bundles on X forms a commutative semiring with cancellation when taking the direct sum $\oplus$ as the additive operation and the tensor product $\otimes$ as the multiplicative operation. This semiring is denoted $J(X)$.

Proving the above claim takes some work, but the full proof is given in Section 4.1 of this chapter. Most of the proof is routine verification, but getting the cancellation property and verifying that multiplication is well-defined appeals to Claim 4.19, which is where the compact Hausdorff condition is used. Because this semiring is commutative with the cancellation property, it is most convenient to consider the unique commutative ring promised through ring completion.

Definition 5.3 (K-Theory of a Topological Object)**.** Take compact Hausdorff base space X and let $J(X)$ denote the commutative semiring with cancellation as described in claim. Then, the ring completion of $J(X)$ is the *K-theory of* X and is denoted $K(X)$.

To get a feel for this definition, consider the following computations of K-theory on simple topological spaces.

Example 5.4 (K-Theory of a point)**.** Consider as a topological space a single point $\{x_0\}$. The only choice of vector bundles on $\{x_0\}$ are the trivial bundles of each dimension. That is, the set $\left\{\varepsilon^0, \varepsilon^1, \varepsilon^2, \dots\right\}$. No two trivial bundles will be in the same stable isomorphism class, giving the set of equivalence classes $\left\{[\varepsilon^0], [\varepsilon^1], [\varepsilon^2], \dots\right\}$, which is isomorphic to the semiring $\mathbb{N}$. So, the ring $K(\{x_0\})$ is the ring completion of $\mathbb{N}$. That is, $K(\{x_0\}) \cong \mathbb{Z}$.

Example 5.5 (K-Theory of n points)**.** Consider the topological space of n disconnected points $\{x_0, x_2, \dots, x_{n-1}\}$. Then, each point can have a fiber of any dimension, and the choice of fibers is independent of one another. This gives that the set of all vector bundles is isomorphic to the set $\mathbb{N}^n$. Then, any arbitrary vector bundle over the space can be denoted $(\varepsilon^{k_0}, \varepsilon^{k_1}, \dots, \varepsilon^{k_{n-1}})$ where the first element in the tuple represents the bundle over the first disconnected point, the second element represents the bundle over the second, and so on. The equivalence classes can then be represented $[(\varepsilon^{k_0}, \varepsilon^{k_1}, \dots, \varepsilon^{k_{n-1}})]$, and in this case every vector bundle is its own equivalence class, and so this is isomorphic to the semiring $\mathbb{N}^n$, which has ring completion $\mathbb{Z}^n$. Thus $K(\{x_0, x_1, \dots, x_{n-1}\}) \cong \mathbb{Z}^n$.

However, defining the K-theory on topological objects only brings the operation half way to being a functor. Functors map objects to objects but also morphisms to morphism. Just as K-theory brings topological spaces to rings, K-theory must bring continuous functions between topological spaces to homomorphisms of rings. In this case, K-theory is a contravariant functor and so reverses the direction of the mapping.

Claim 5.6. Take topological spaces X and Y with a continuous function $f : X \to Y$. Let $J(X)$ denote the semiring as described in claim 5.2 and let $K(Y)$ be the K-theory of Y. Further, define the function $J(f) : J(X) \to K(Y)$ defined on an equivalence class $[E] \in J(X)$ by

$$J(f) : [E] \mapsto [f^*(E)]$$

where f^* denotes the pullback. Then, $J(f)$ is a well-defined homomorphisms of semirings.

Verifying the above follows easily from the properties of pullback given in Claim 4.16. The full proof is given in section 4.2. Note that this is the point where the contravariant property emerges. Because the elements of the semiring is equivalence classes of vector bundles, a homomorphism consists of mappings from one vector bundle to another vector bundle. This is best done through the induced vector bundle by the pullback which, as discussed previously, must be done in the reverse direction.

Definition 5.7 (K-Theory of a Topological Morphism)**.** For compact Hausdorff spaces X and Y with a continuous function $f : X \to Y$, let $J(f) : J(X) \to J(Y)$ denote the homomorphism of semirings as described in Claim 5.6. Further, let $i_X : J(X) \to K(X)$ and $i_Y : J(Y) \to K(Y)$ denote ring completions. Then, the *K-theory of f* is the unique homomorphism of rings $K(f) : K(X) \to K(Y)$ such that the following diagram commutes as promised by the universal property. That is, that $K(f) \circ i_X = i_Y \circ J(f)$.

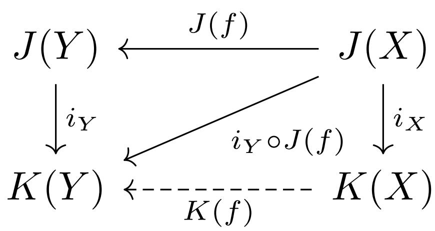

FIGURE 1. Definition of $K(f)$ through Universal Property

And that is the definition of K-theory! Figure 2 denotes a diagram of the K-theory functor. Next, observe how the K-theory functor on the following examples of concrete topological spaces.

Example 5.8. This example examines the K-theory of the inclusion from the space with one point to the space with n points. So take the topological space of n points $\{x_0, x_1, \ldots, x_{n-1}\}$ and consider the subspace $\{x_0\} \subset \{x_0, \ldots, x_{n-1}\}$. Then, let $i : \{x_0\} \to \{x_0, \ldots, x_{n-1}\}$ be the inclusion map.

Recall that the equivalence classes of vector bundles over x_0 can be represented $[\varepsilon^n]$ and the equivalence classes over $\{x_0, x_1, \ldots, x_{n-1}\}$ can be represented $[(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})]$. Then, by definition the function $J(i) : J(\{x_0, \ldots, x_{n-1}\}) \to J(\{x_0\})$ is given by

$$J(i) : [(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})] \mapsto [i^*(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})]$$

Recall that the pullback of the inclusion is simply the restriction to the space in the domain. In this case, that is the restriction to the point x_0 and so function $J(i)$ is

$$J(i) : [(\varepsilon^{k_0}, \varepsilon^{k_1}, \ldots, \varepsilon^{k_{n-1}})] \mapsto [\varepsilon^{k_0}]$$

Now step away from the vector bundles themselves and let $J(i)$ denote the semiring homomorphism $J(i) : \mathbb{N}^n \to \mathbb{N}$ given by $J(i) : (k_0, k_1, \ldots, k_{n-1}) \mapsto k_0$. Finally, consider the ring completions $\mathbb{Z}^n$

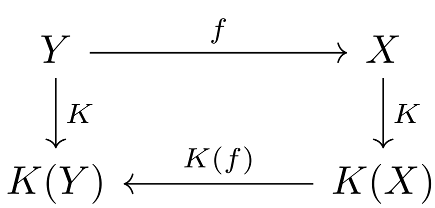

FIGURE 2. The K-Theory Functor

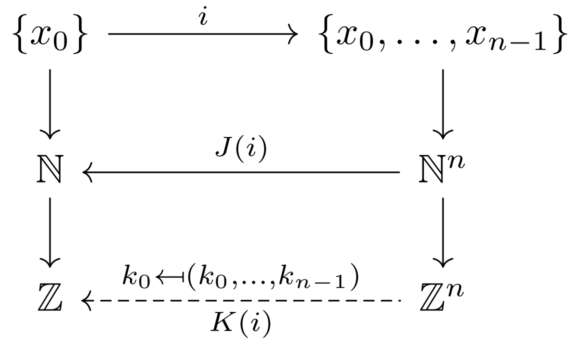

FIGURE 3. K-Theory of a Morphism Example

and $\mathbb{Z}$. The universal property then promises a unique ring homomorphism $K(i) : \mathbb{Z}^n \to \mathbb{Z}$. In this case, the ring homomorphism is given by $K(i) : (k_0, k_1, \ldots, k_{n-1}) \mapsto k_0$, but this time, each k_i can take on the value of any integer. See figure 3 for a visual of this construction.

2. The Reduced K-Theory Functor $\widetilde{\mathbf{K}}$

There is another closely related vein of K-theory called *reduced K-theory.* Reduced K-theory is a is a functor from the category of topological spaces to the category of abelian groups. However, with as assumption discussed later, this functor can be extended to the category of commutative rings (but not necessarily with identity). Reduced K-theory uses a stronger equivalence relation, which gives fewer equivalence classes.

Definition 5.9. Define the equivalence relation $\sim$ on vector bundles E_1 and E_2 over the same base space such that $E_1 \sim E_2$ if $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^m$ for some n and m.

Then, this equivalence class immediately gives rise to the desired group.

Definition 5.10. Take a compact Hausdorff topological space X and let $\widetilde{K}(X)$ denote the set of all equivalence classes under the relation $\sim$ as described in definition 5.9. Then, define the group operation by the direct sum $\oplus$ operation on the elements. This forms a well-defined abelian group and is called the *reduded K-theory* of X.

The verification that $\widetilde{K}(X)$ indeed forms a well-defined group is straight-forward, but it is worth noting that the existence of inverses uses Claim 4.19, which requires the compact Hausdorff condition.

Consider some simple computation of reduced K-theory.

Example 5.11 (Reduced K-Theory of a Point)**.** Again as a topological space a single point $\{x_0\}$. The only choice of vector bundles on $\{x_0\}$ is the set trivial bundles $\left\{\varepsilon^0, \varepsilon^1, \varepsilon^2, \ldots\right\}$. In this case, however, each trivial bundle is in the same isomorphism class, so the set of equivalence classes has only the identity element ε^0. So, the reduced K-theory of a point $\widetilde{K}(\{x_0\})$ is the trivial group.

It must still be addressed how reduced K-theory maps continuous topological maps to group morphisms, which again makes use of the pullback.

Definition 5.12 (Reduced K-Theory of a Topological Morphism)**.** Let $f : Y \to X$ denote a continuous function between topological spaces. Then the induced mapping $\widetilde{K}(f) : \widetilde{K}(X) \to \widetilde{K}(Y)$ is defined by

$$\widetilde{K}(f) : [E] \mapsto [f^*(E)]$$

where the equivalence classes are with respect to the relation $\sim$ as in definition 5.9.

The verification for well-defined is identical to that for unreduced K-theory. With well-defined, the properties of pullback $\mathrm{Id}^*(E) \approx E$ and $(f \circ g)^*(E) \approx g^*(f^*(E))$ immediately give that $\widetilde{K}$ obeys the rules for functors.

Now consider the following example, which demonstrates an important relationship between the functors K and $\widetilde{K}$.

Example 5.13 (Reduced K-Theory of n points)**.** Consider the topological space of n disconnected points $\{x_0, x_2, \dots, x_{n-1}\}$. Again, each fiber is free to have a vector space of any dimension, so any arbitrary vector bundle over the space can be denoted $(\varepsilon^{k_0}, \varepsilon^{k_1}, \dots, \varepsilon^{k_{n-1}})$. The equivalence classes under $\sim$ are then represented $[(\varepsilon^{k_0}, \varepsilon^{k_1}, \dots, \varepsilon^{k_{n-1}})]_\sim$. In this case, every equivalence class has more than one element. In particular, $(\varepsilon^{k_0}, \varepsilon^{k_1}, \dots, \varepsilon^{k_{n-1}}) \sim (\varepsilon^{l_0}, \varepsilon^{l_1}, \dots, \varepsilon^{l_{n-1}})$ if there exists bundles ε^k and ε^l such that

$$(\varepsilon^{k_0} \oplus \varepsilon^k, \varepsilon^{k_1} \oplus \varepsilon^k, \dots, \varepsilon^{k_{n-1}} \oplus \varepsilon^k) \approx (\varepsilon^{l_0} \oplus \varepsilon^l, \varepsilon^{l_1} \oplus \varepsilon^l, \dots, \varepsilon^{l_{n-1}} \oplus \varepsilon^k)$$

By simply ditching the "ε" symbol, there is an isomorphism to equivalence classes of n tuples of integers $[(k_0, k_1, \dots, k_{n-1})]$ where $(k_0, k_1, \dots, k_{n-1}) \sim (l_0, l_1, \dots, l_{n-1})$ if there exists integers k and l such that

$$(k_0 + k, k_1 + k, \dots, k_{n-1} + k) = (l_0 + l, l_1 + l, \dots, l_{n-1} + l)$$

Additionally, the group operation is element wise addition as is taken from the representation with "ε". Note that all of the k's and l's are allowed to be any integer but they originally represented the dimension of a trivial bundle, so it seems they should only be nonnegative integers. However, expanding the elements to integers does not change the group, for every new element containing a negative integer will land in a preexisting equivalence class.

And so, elements in reduced K-theory are of the form of equivalence classes $[(k_0, k_1, \dots, k_{n-1})]$ with the relation as defined earlier.

However, this does not clear up what this reduced K-theory is isomorphic to. The bottom line is that the reduced K-theory of n points is isomorphic to the group $\mathbb{Z}^{n-1}$. There are a few ways to see this, but the most educational is with the following.

Fix the point $x_0 \in \{x_0, \dots, x_{n-1}\}$ and consider the K-theory groups $K(\{x_0, \dots, x_{n-1}\})$ and $K(\{x_0\})$. The goal will be to construct a homomorphism $\varphi : \widetilde{K}(\{x_0, \dots, x_{n-1}\}) \to K(\{x_0, \dots, x_{n-1}\})$ and use φ with the mapping $K(i) : K(\{x_0, \dots, x_{n-1}\})$ as defined previously. Overall, this will give the chain of mappings as shown in figure 4. Additionally takes note of the isomorphisms $K(\{x_0, \dots, x_{n-1}\}) \cong \mathbb{Z}^n$ and $K(\{x_0\}) \cong \mathbb{Z}$.

Now, define φ on the discussed representations on the K-theory and reduced K-theory of n points as follows.

$$\varphi : [(k_0, k_1, \dots, k_{n-1})] \to (0, k_1 - k_0, \dots, k_{n-1} - k_0)$$

The mapping φ is indeed a group homomorphism and now recall that $K(i)$ is given by the following.

$$K(i) : (l_0, l_1, \dots, l_{n-1}) \mapsto l_0$$

Make a two observations. Firstly, $\mathrm{Im}(\varphi) = \mathrm{Ker}(K(i))$. Secondly, φ is injective. Then φ injective gives $\widetilde{K}(\{x_0, \dots, x_{n-1}\}) \cong \mathrm{Im}(\varphi)$ which then gives the relationship

$$\widetilde{K}(\{x_0, \dots, x_{n-1}\}) \cong \mathrm{Ker}(K(i))$$

Lastly note that element of the kernel are of the form $(0, l_1, \dots, l_{n-1})$ for any choice of l's, so the kernel is isomorphic to $\mathbb{Z}^{n-1}$. Overall, this gives $\widetilde{K}(\{x_0, \dots, x_{n-1}\}) \cong \mathbb{Z}^{n-1}$.

$$\widetilde{K}(\{x_0, \dots, x_{n-1}\}) \xrightarrow{\varphi} K(\{x_0, \dots, x_{n-1}\}) \xrightarrow{K(i)} K(\{x_0\})$$

Figure 4. Chain of Homomorphisms for n-points example

The above example found the reduced K-theory by demonstrating that $\widetilde{K}(\{x_0, \dots, x_{n-1}\})$ is isomorphic to $\mathrm{Ker}(K(i))$ In fact, a relationship like this exists in general. For any topological space X with point $x_0 \in X$ and inclusion $i : x_0 \to X$, the relationship $\widetilde{K}(X) \cong \mathrm{Ker}(K(i))$ holds. The proof of this uses tools developed in the next chapter, but the above example gives a taste of the proof. A consequence of this, however, is that $K(i)$ is a ring homomorphism, so $\mathrm{Ker}(K(i))$ is an ideal. Then $\widetilde{K}(X)$ is isomorphic to this ideal and thus can be given a multiplication. Thus, we can consider $\widetilde{K}(X)$ to be a ring, but not necessarily with identity. The ring structure, however, does depend on the point x_0, thus to make the choice not arbitrary, X must be a *pointed* topological space, which associates a pre-determined point x_0 to the space. In symmetrical spaces such as the torus and the sphere, every choice of point is equivalent, so this distinction is not always necessary.

The computed examples hint at another relationship between K-theory and reduced K-theory. Note that for a point $\{x_0\}$, $K(\{x_0\}) \cong \mathbb{Z}$ and $\widetilde{K}(\{x_0\}) \cong \{0\}$. The computations for a collection of points $\{x_0, x_1, \dots, x_{n-1}\}$ showed $K(\{x_0, x_1, \dots, x_{n-1}\}) \cong \mathbb{Z}^n$ and $\widetilde{K}(\{x_0, x_1, \dots, x_{n-1}\}) \cong \mathbb{Z}^{n-1}$. Note that $\mathbb{Z} \cong \{0\} \oplus Z$ and more generally, $\mathbb{Z}^n \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}$. More generally, it is true that $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$ for any topological space X, but this proof again uses techniques of the following chapter.

3. Simplifying the Notation

Reduced K-theory does not require any ring extension, so an element of some reduced K-theory group $\widetilde{K}(X)$ is some equivalence class $[E]$ with representative E. However, often the equivalence class is dropped so an element of $\widetilde{K}(X)$ can be written as the bundle E. Embracing ring notation, the sum of two bundles can be written by $E_1 + E_2$, but note that this addition is defined to be the direct sum. The differences such of two bundles can be written $E_2 - E_1$, the repeated addition of a bundle n times can be written $n \cdot E$, and the additive identity, which represents the trivial bundle of any dimension, can be denoted by 0.

However, now consider the elements of a non-reduced K-theory group $K(X)$. Note that not every element of $K(X)$ is simply an equivalence class $[E]$, for this only forms the semiring. Understanding the form of the whole ring requires an examination of the construction of the ring completion. As discussed previously, the ring is constructed as equivalence classes on two elements, understood to represent a difference, just as the integers can be constructed from the nonnegative integers by considering all differences $n_1 - n_2$. A good way to represent this equivalence class on two elements is then by a subtraction sign between two equivalence classes $[E_1] - [E_2]$. Again, the equivalence class notation can be dropped and so elements of $K(X)$ can be represented in the notation $E_1 - E_2$ for two bundles E_1 and E_2; however, in the special case that the bundle E_2 is 0 in the difference, the bundle E_1 can be considered as an element of $K(X)$ directly. Adding together two elements in $K(X)$ can be denoted by $(E_1 - F_1) + (E_2 - F_2)$, which is equivalent to $(E_1 + E_2) - (F_1 + F_2)$ as expected. The additive identity element ε^0 can be represented by 0, and the multiplicative identity ε^1 can be represented by 1. In fact, any trivial bundle ε^n can be represented by n.

Example 5.14. Take the base space to be $\mathbb{C}P^1$ (which is isomorphic to S^2), and let H denote the canonical line bundle. Then the difference $H - 1$ can be considered an element of $K(X)$, and of course the identity 1 is an element of $K(X)$. In fact, any repeated sum of these two elements is in $K(X)$ and can be represented by $n + m(H - 1)$.

Given some continuous function $f : X \to Y$, the induced homomorphism between K rings is written $K(f) : K(Y) \to K(X)$ and the induced homomorphism between $\widetilde{K}$ rings is written $\widetilde{K}(f) : \widetilde{K}(Y) \to \widetilde{K}(X)$. However, decorating every function with K and $\widetilde{K}$ becomes cramped when considering long compositions of induced homomorphisms, so the notation is simplified to letting f^* denote either $K(f)$ or $\widetilde{K}(f)$ depending on the context.

4. Verifications

4.1. Semiring Verification.

Proof. Take a compact Hausdorff space X. It must be verified that the set of stable isomorphism classes of vector bundles over X with operations defined by the direct sum $\oplus$ and the tensor product $\otimes$ indeed satisfies all the properties of a commutative semiring with additive cancellation.

Before proceeding further, it must be verified that addition is well defined. So, take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X. Then, take nonnegative integers n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition. Then it follows that $E_1 \oplus F_1 \approx_s E_2 \oplus F_2$ by the following chain of equalities.

$$(E_1 \oplus F_1) \oplus \varepsilon^{n+m} \approx (E_1 \oplus \varepsilon^n) \oplus (F_1 \oplus \varepsilon^m) \approx (E_2 \oplus \varepsilon^n) \oplus (F_2 \oplus \varepsilon^m) \approx (E_2 \oplus F_2) \oplus \varepsilon^{n+m}$$

Where the equivalence $\varepsilon^{n+m} \approx \varepsilon^n \oplus \varepsilon^m$ was used.

With addition well defined, the associativity and commutativity of addition follows directly from the associativity and commutativity of the direct sum on vector bundles. Further, the result $E \oplus \varepsilon^0 \cong E$ for any vector bundle E stated in Claim 4.19 makes the equivalence class $[\varepsilon^0]$ the additive identity.

The additive cancellation follows from Claim 4.19, which applies here by X compact Hausdorff. Indeed, take bundles E, F, and S over X such that $[E] + [S] = [F] + [S]$. First note that in the case of S trivial, $[E] = [F]$ by definition. Otherwise, Claim 4.19 promises a bundle S' such that $S \oplus S'$ is trivial. Adding $[S']$ to both sides reduces the expression to the first case with $[E] + [S \oplus S'] = [F] + [S \oplus S']$, giving $[E] = [F]$ as desired.

Before proceeding with any multiplicative verifications, it must be verified that the tensor product $\otimes$ gives a well defined multiplicative operation. So, again take $E_1 \approx_s E_2$ and $F_1 \approx_s F_2$ to be vector bundles over X and nonnegative integers n and m such that $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^n$ and $E_1 \oplus \varepsilon^m = E_2 \oplus \varepsilon^m$ as promised by definition. Next, define the bundle M by

$$M \approx \varepsilon^n \otimes (F_1 \oplus \varepsilon^m) \oplus \varepsilon^m \otimes (E_1 \oplus \varepsilon^n) \approx \varepsilon^n \otimes (F_2 \oplus \varepsilon^m) \oplus \varepsilon^m \otimes (E_2 \oplus \varepsilon^n)$$

Next, observe that M is constructed exactly so that the relation $E_1 F_1 \oplus M \approx E_2 F_2 \oplus M$ holds:

$$E_1 \otimes F_1 \oplus M \approx (E_1 \oplus \varepsilon^n)(F_1 \oplus \varepsilon^m) \oplus \varepsilon^n \varepsilon^m \approx (E_2 \oplus \varepsilon^n)(F_2 \oplus \varepsilon^m) \oplus \varepsilon^n \varepsilon^m \approx E_2 \otimes F_2 \oplus M$$

So, take M' to be the bundle such that $M \oplus M'$ is trivial as promised by Claim 4.19. Then, the desired conclusion follows easily, giving that multiplication is well-defined.

$$E_1 \otimes F_1 \oplus (M \oplus M') = E_2 \otimes F_2 \oplus (M \oplus M')$$

With multiplication well defined, the associativity and commutativity of multiplication follows directly from the associativity and commutativity of the tensor product on vector bundles. Similarly, the distributivity of $\otimes$ over $\oplus$ in vector bundles gives that the defined multiplication distributes over the defined addition. Finally, the result $E \otimes \varepsilon^1 \cong E$ for any vector bundle E makes the equivalence class $[\varepsilon^1]$ the multiplicative identity.

□

4.2. Homomorphism of Semirings Verification.

Proof. Let $f : X \to Y$ denote a continuous function between two compact Hausdorff spaces.

First it must be verified that $J(f)$ is well-defined. Specifically, it must be shown that if $[E_1] = [E_2]$, then $J(f)([E_1]) = J(f)([E_2])$. That is, it must be shown that $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^n$ for some n implies $f^*(E_1) \approx_s f^*(E_2)$. First, note the following application of the distributivity of pullback over direct sum taken from Claim 4.16

$$f^*(E_1) \oplus f^*(\varepsilon^n) \approx f^*(E_1 \oplus \varepsilon^n) \approx f^*(E_1 \oplus \varepsilon^n) \approx f^*(E_2) \oplus f^*(\varepsilon^n)$$

The result that the pullback of a trivial bundle is trivial combined with the above confirms $f^*(E_1) \approx_s f^*(E_2)$ and so $J(f)$ is well-defined.

With $J(f)$ well-defined, verifying that $J(f)$ is a semiring homomorphism follows easily from the properties of pullback. Specifically, the distributivity of pullback over direct sum directly gives the distributivity of $J(f)$ over the defined addition. Similarly, the distributivity of pullback over tensor product gives that $J(f)$ distributes over the defined multiplication. Lastly, the property that f^* maps the bundle ε^1 over X to the bundle ε^1 over Y implies that $J(f)$ maps the multiplicative identity to the multiplicative identity. □

4.3. K-Theory Functor Satisfies Contravariant Composition Law.

PROOF. Let X, Y, and Z denote compact Hausdorff spaces and take $f : X \to Y$ and $g : Y \to Z$ be continuous functions between them.

Further denote the semirings as defined in Claim 5.2 by $J(X)$, $J(Y)$, and $J(Z)$. Additionally, let $i_X : J(X) \to K(X)$, $i_Y : J(Y) \to K(Y)$, and $i_Z : J(Z) \to K(Z)$ denote the ring completions of each semiring as in definition 5.3. Further, let $J(g) : J(Z) \to J(Y)$ and $J(f) : J(Y) \to J(X)$ denote the homomorphism of semirings as described in Claim 5.6. Finally, let the functions $K(g) : K(Z) \to K(Y)$, $K(f) : K(Y) \to K(X)$, and $K(f \circ g) : K(Z) \to K(X)$ be the unique functions such that the following composition identities hold.

$$K(f) \circ i_X = i_Y \circ J(f)$$
$$K(g) \circ i_Y = i_Z \circ J(g)$$
$$K(f \circ g) \circ i_X = i_Z \circ J(f \circ g)$$

Additionally note that the relation $J(f \circ g) = J(g) \circ J(f)$ follows from the discussed pullback property $(f \circ g)^*(E) = g^*(f^*(E))$ on a bundle E by the following computation on an element $[E] \in J(X)$.

$$J(f \circ g)([E]) = [(f \circ g)^*(E)] = [g^*(f^*(E))] = J(g)([f^*(E)]) = J(g)(J(f)([E]))$$

Substitutions of the preceding result together with the earlier composition identities allows for the following result.

$$(K(g) \circ K(f)) \circ i_X = K(g) \circ (i_Y \circ J(f)) = (i_Z \circ J(Y)) \circ J(f) = i_Z \circ J(f \circ g)$$

And so $K(f) \circ K(g)$ fulfills the defining property of $K(f \circ g)$. Because the function $K(f \circ g)$ is the unique function fulfilling this property, it must be that $K(f) \circ K(g) = K(f \circ g)$. Figure 5 provides a visual aid for this argument.

□

4.4. Reduced K-Theory forms Group.

PROOF. First it must be verified that the direct sum operation $\oplus$ is well-defined on the equivalence classes. So, consider vector bundles $E_1 \sim E_2$ and $F_1 \sim F_2$. Then let n_1, m_1, n_2, m_2 be the numbers such that $E_1 \oplus \varepsilon^{n_1} \approx E_2 \oplus \varepsilon^{n_2}$ and $F_1 \oplus \varepsilon^{m_1} \approx F_2 \oplus \varepsilon^{m_2}$. It then follows that $E_1 \oplus F_1 \approx E_2 \oplus F_2$

$$(E_1 \oplus F_1) \oplus (\varepsilon^{n_1+m_1}) \approx (E_1 \oplus \varepsilon^{n_1}) \oplus (F_1 \oplus \varepsilon^{m_1}) \approx (E_2 \oplus \varepsilon^{n_2}) \oplus (F_2 \oplus \varepsilon^{m_2}) \approx (E_1 \oplus F_1) \oplus (\varepsilon^{n_2+m_2})$$

Where the above computation used $\varepsilon^{n+m} \approx \varepsilon^n \oplus \varepsilon^m$.

With the group operation well-defined, the associativity and commutativity of the operation follows from direct sum associative and commutative on bundles.

The identity element in the group is given by the equivalence class $[\varepsilon^0]$, which is the set of all trivial bundles. Indeed, $[E] + [\varepsilon^0] = [E \oplus \varepsilon^0] = [E]$.

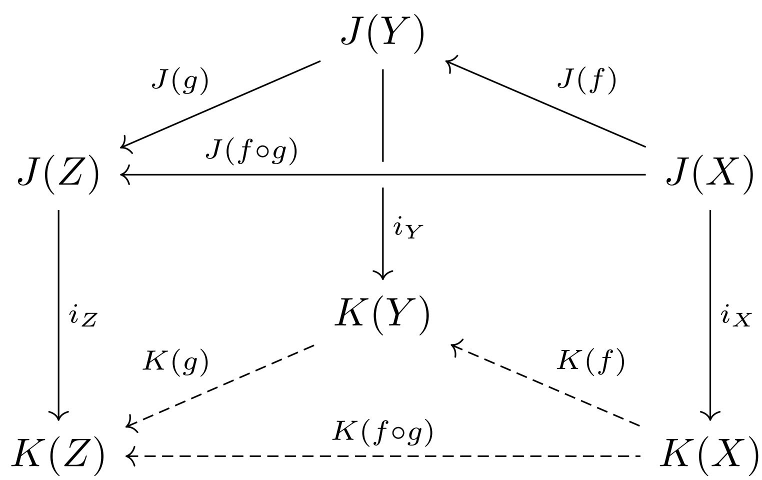

Figure 5. K-Theory Contravariant Composition Argument

It only remains to show the existence of inverses, which uses Claim 4.19. Then take any element $[E]$ and consider the promised bundle E' such that $E \oplus E' \approx \varepsilon^n$ for some trivial bundle of dimension n. Then, the element $[E']$ is the inverse element.

$$[E] + [E'] = [E \oplus E'] = [\varepsilon^n] = [\varepsilon^0]$$

□

CHAPTER 6

K-theory as a Cohomology Theory

The functors K and $\widetilde{K}$ take the category of topological spaces to the category of rings. However, K-theory requires extra patterns in the induced rings to be useful. The patterns of K-theory are particularly bountiful when taking the base space to be a sphere; for instance, $\widetilde{K}(S^2n) = \widetilde{K}(S^2)$ for any n. These patterns allow for a space-subspace pair $A \subset X$ to induce not just one ring, but an infinite sequence of groups with pleasant patterns. The construction of such an infinite sequence is a *cohomology theory*, which holds useful information about the original space-subspace pair. The *external product* is an important structure of K-theory that is responsible for the connection to spheres and the application in the following chapter.

1. Exact Sequences

The present goal is to construct an infinite sequence of groups with pleasant properties. The following definition descries one such property.

Definition 6.1 (Short Exact Sequence)**.** Let A, B, and C be abelian groups. Then let $\psi : A \to B$ be an injective homomorphism and let $\varphi : B \to C$ be a surjective homomorphism. Further suppose that $\operatorname{Ker}\varphi = \operatorname{Im}\psi$. Then, the pair of homomorphisms ψ and φ are called *exact* and the sequence $A \xrightarrow{\psi} B \xrightarrow{\varphi} C$ is called a *short exact sequence*.

Note that an exact sequence does not need to be short. A sequence of arbitrary length can *exact* so long as each adjacent pair of homomorphisms is exact. Of particular note is the *long exact sequence*, which is a sequence of groups extending infinitely in both directions such that each adjacent pair of homomorphism is exact.

$$\ldots \to A^{-2} \xrightarrow{\alpha} A^{-1} \xrightarrow{\beta} A^{0} \xrightarrow{\gamma} A^{1} \xrightarrow{\delta} A^{2} \to \ldots$$

The above definitions of exact sequences used groups, but replacing the word "groups" with "commutative rings" or even with "modules" gives a definition of exact sequence for other categories.

Exact sequences are useful. Recall that the computation of the reduced K-theory for n points in example 5.13 makes use of an exact sequence. The following result is partially responsible for why exact sequences are so useful.

Lemma 6.2 (Splitting Lemma)**.** Take a short exact sequence $A \xrightarrow{\psi} B \xrightarrow{\varphi} C$. Then, the following three statements are equivalent:

(i) There exists a homomorphism $\alpha : B \to A$ such that $\alpha \circ \psi$ is the identity on A.
(ii) There exists a homomorphism $\beta : C \to B$ such that $\varphi \circ \beta$ is the identity on C.

(iii) There is an isomorphism $A \oplus C \cong B$.

Further note that the resulting homomorphisms α and β are exact.

A proof of this lemma is omitted from this document, but [**2**] a complete discussion on short exact sequences.

Example 6.3. Let X be a compact Hausdorff pointed topological space, and let $x_0 \in X$ be the particular point of the pointed topological space. Now define $q^* : K(X) \to \widetilde{K}(X)$ be given by the completion of the mapping $q^* : (E_1 - E_2) \to (E_1 - E_2)$. Note that this is well-defined because the equivalence classes of $\widetilde{K}(X)$ are only coarser equivalence classes of $K(X)$. Next, let $r : X \to x_0$ take every $x \in X$ to the point x_0, and let $r^* : K(X) \to K(\{x_0\})$ represent the induced homomorphism of K-rings. This gives the following short sequence of homomorphisms.

$$K(\{x_0\}) \xrightarrow{r^*} K(X) \xrightarrow{q^*} \widetilde{K}(X)$$

To see this is an exact sequence, note that any element of $K(\{x_0\})$ can be written as the difference of two trivial bundles $\varepsilon^{n_1} - \varepsilon^{n_2}$. But then $r(\varepsilon^{n_1} - \varepsilon^{n_2})$ will be given by the pullback $r^*(\varepsilon^{n_1}) - r^*(\varepsilon^{n_2})$, which will simply be the element $\varepsilon^{n_1} - \varepsilon^{n_2}$ over base space X. However, note that $q^* : (\varepsilon^{n_1} - \varepsilon^{n_2})$ is equivalent to ε^0 under the equivalence relation, and thus $q^* r^* = 0$, confirming $\mathrm{Ker}(q^*) \subset \mathrm{Im}(r^*)$. For the other direction, note that the set of all trivial bundles is the equivalence class representing 0 in $\widetilde{K}(X)$, so an $\mathrm{Ker}(q^*)$ must be of the form $\varepsilon^{n_1} - \varepsilon^{n_2}$, which has preimage $\varepsilon^{n_1} - \varepsilon^{n_2}$ in $K(\{x_0\})$, thus $\mathrm{Im}(r^*) \subset \mathrm{Ker}(q^*)$. Thus the sequence is indeed exact.

Now take the inclusion $i : x_0 \to X$ and let $i^* : K(X) \to K(\{x_0\})$ be the induced homomorphism of K-rings. Note that $r \circ i$ is the identity on x_0, so by the factorial properties of K, this gives $i^* \circ r^*$ is the identity on $K(\{x_0\})$, which fulfills statement (i) in Lemma 6.2. This then implies the other two statements in the lemma, which correspond to the following two results.

Claim 6.4. Let i^* and each term in the exact sequence $K(\{x_0\}) \xrightarrow{r^*} K(X) \xrightarrow{q^*} \widetilde{K}(X)$ be defined in Example 6.3. Then $K(X) \cong \widetilde{K}(X) \oplus K(\{x_0\})$.

Proof. As shown in Example 6.3, the exact sequence in the claim satisfies property i of the Splitting Lemma, which gives statement (iii), which directly gives $K(X) \cong \widetilde{K}(X) \oplus K(\{x_0\})$, and by Example 5.4, $K(\{x_0\}) \cong \mathbb{Z}$, so this can be rewritten $K(X) \cong \widetilde{K}(X) \oplus K(\{x_0\})$. □

Claim 6.5. Let i^* and each term in the sequence $K(\{x_0\}) \xrightarrow{r^*} K(X) \xrightarrow{q^*} \widetilde{K}(X)$ be defined in Example 6.3. Then $\widetilde{K}(X) \cong \mathrm{Ker}(i^*)$.

Proof. As shown in Example 6.3, the exact sequence in the claim satisfies property i of the Splitting Lemma, which gives property (ii). Let $j^* : \widetilde{K}(X) \to K(X)$ be the promised homomorphism such that fulfills $q^* \circ j^*$ is the identity on $\widetilde{K}(X)$. This forces j^* to be injective, so $\mathrm{Im}(j^*) \cong \widetilde{K}(X)$. But by j^* and i^* exact, $\widetilde{K}(X) \cong \mathrm{Ker}(i^*)$. □

2. Exact Sequences in K-theory

However, exact sequences has a bigger role in K-theory than highlighting the relationship between K-theory and reduced K-theory. Given a pair (X, A) of topological spaces with particular properties, the K-theory functor induces a natural long exact sequence of groups with nice properties. This mapping from a pair (X, A) to a long exact sequence is the idea of *cohomology*. This begins with the following short exact sequence construction.

Take the pair of compact Hausdorff topological spaces (X, A) with $A \subset X$ to a closed, contractible subset. Then consider the inclusion and quotient maps $i : A \to X$ with $q : X \to X/A$. This can also be written $A \xrightarrow{i} X \xrightarrow{q} X/A$. This sequence of morphisms between topological spaces induces a sequence of morphisms between commutative rings (possibly without identity) $\widetilde{K}(X/A) \xrightarrow{q^*} \widetilde{K}(A) \xrightarrow{i^*} \widetilde{K}(X)$ as depicted in Figure 1. Further the sequence of ring homomorphisms is exact, which is proved in [**4**].

$$\begin{array}{ccccc} A & \xrightarrow{\quad i \quad} & X & \xrightarrow{\quad q \quad} & X/A \\ \downarrow{\scriptstyle \widetilde{K}} & & \downarrow{\scriptstyle \widetilde{K}} & & \downarrow{\scriptstyle \widetilde{K}} \\ \widetilde{K}(A) & \xleftarrow{\quad i^* \quad} & \widetilde{K}(X) & \xleftarrow{\quad q^* \quad} & \widetilde{K}(X/A) \end{array}$$

Figure 1. Inducing Sequence of Rings

Note that the map r in Example 6.3 composed to the identity with the inclusion map, $r \circ i = \mathrm{Id}$ allowed for the convenient application of the Splitting Lemma. In fact, any map r that satisfies this property is called a *retract*, and the existence of a retract will guarantee a splitting. Note the following exmaple.

Example 6.6. Consider the exact sequence depicted in Figure 1 with the space $X \vee Y$ with subspace $Y \subset X \vee Y$ where Y is contractible. Further note that $(X \vee Y)/Y \approx X$. This gives the exact sequence

$$\widetilde{K}(X) \xrightarrow{q^*} \widetilde{K}(X \vee Y) \xrightarrow{i^*} \widetilde{K}(Y)$$

Additionally, there is a retract $r : X \vee Y \to Y$, for Y is contractible and thus can be collapsed to point. Then $i^* \circ r^* = \mathrm{Id}$ allows for an application of the Splitting Lemma, which gives $\widetilde{K}(X \vee Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y)$.

This short exact sequence can be extended into a slightly longer exact sequence. First consider the sequence of spaces beginning with A and X such that each additional space is given by the disjoint union of the previous space together with the cone of the space two steps back. The disjoint unions allow for natural inclusions between spaces as depicted in the following diagram.

$$A \xrightarrow{i} X \xrightarrow{i} X \cup CA \xrightarrow{i} (X \cup CA) \cup CX \xrightarrow{i} ((X \cup CA) \cup CX) \cup C(X \cup CA)$$

The strategy will be to apply claim **??** to show that each adjacent pair of ring homomorphisms induced by $\widetilde{K}$ is exact. However, applying this claim requires taking the topological quotient of each space by collapsing the subspace two steps back to a point.

$$X \xrightarrow{q} X/A \qquad X \cup CA \xrightarrow{q} (X \cup CA)/CX \qquad (X \cup CA) \cup CX \xrightarrow{q} ((X \cup CA) \cup CX)/(CX)$$

So, the above diagrams define the inclusion maps and show the necessary quotient maps to incorporate, but claim **??** requires a relationship between the inclusion and quotient maps that is currently missing from this construction. For this, consider the relationship between spaces given by the following additional quotient maps, which result from quotienting out the by the most recently added cone space.

$$X \cup CA \xrightarrow{Q} X/A \qquad\qquad (X \cup CA) \cup CX \xrightarrow{Q} (X \cup CA)/CX$$

$$((X \cup CA) \cup CX) \cup C(X \cup CA) \xrightarrow{Q} ((X \cup CA) \cup CX)/(CX)$$

The following result comes to the rescue.

Claim 6.7. If A is contractible, then the quotient map $q : X \to X/A$ gives an isomorphism $\widetilde{K}(X) \cong \widetilde{K}(X/A)$.

The proof of this is given in [**4**], but the idea is that collapsing a contractible subspace is given a homotopy equivalence, so they will share similar properties — including the $\widetilde{K}$ groups.

The top two rows of Figure 2 summarizes all of the maps discussed so far.

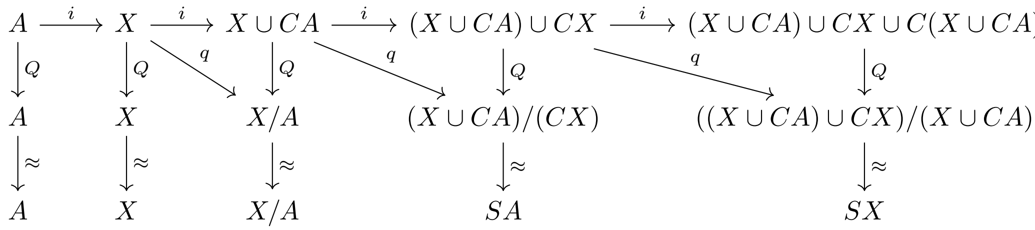

Figure 2. Relationship between Inclusion and Quotient Maps

Figure 2 also includes the following additional pleasant isomorphisms discussed previously in Example 3.29.

$$(X \cup CA)/CX \approx SA \qquad (((X \cup CA) \cup CX) \cup CX) \cup C(X \cup CA))/C(X \cup CA) \approx SX$$

Next, applying the $\widetilde{K}$ functor on all of these maps will take the inclusion maps and quotient maps to ring homomorphisms in the opposite directions. Additionally, $\widetilde{K}$ will take the isomorphisms to ring isomorphisms and will additionally take each map denoted with a "Q" to a ring isomorphism by claim 6.7. This is summarized in Figure 3. Note that each i^*q^* composition gives a short

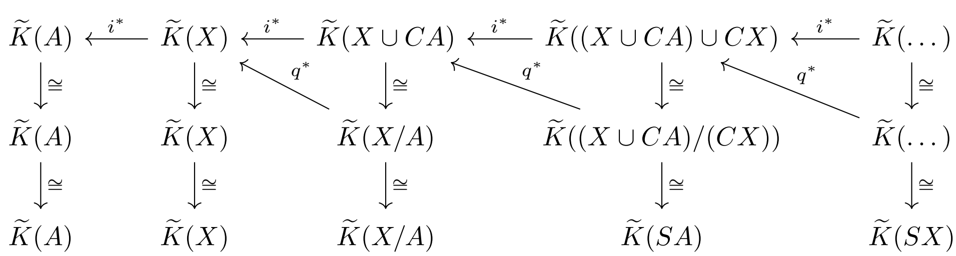

Figure 3. Applying the Functor $\widetilde{K}$

exact sequence as discussed earlier. Then, by viewing each inclusion map as the composition of an isomorphism with a quotient map, the entire top row is exact. By further composing with the vertical isomorphisms provides a simple short exact sequence as shown in figure 4. However, while this sequence is useful, a cohomology theory requires more structure.

$$\widetilde{K}(SX) \longrightarrow \widetilde{K}(SA) \longrightarrow \widetilde{K}(X/A) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(A)$$

FIGURE 4. The Resulting Exact Sequence

3. K-theory as a Cohomology Theory

A cohomology theory requires an infinite collection of functors where each functor takes a fixed subcategory topological space pairs to the category of abelian groups. K-theory considers the subcategory of compact Hausdorff space-subspace pairs — all pairs of the form (A, X) discussed earlier. A cohomology theory must further satisfy the Eilenberg-Steenrod cohomology axioms to some degree. But at a minimum, the functors must take homotopic continuous maps to equivalent homomorphisms, and additionally induce an exact sequence extending infinitely in both directions. A cohomology theory that satisfies these minimum requirements is called a *reduced cohomology theory*. By building upon the work in the previous section, reduced K-theory can be extended into a reduced cohomology theory.

The first step to develop this cohomology theory is to extend the exact sequence developed in the previous section into an exact sequence that extends infinitely in both directions. Figure 4 induces a slightly longer exact sequence given the initial compact Hausdorff space-subspace pair (X, A), but repeating the same process for the pair (SX, SA) will give a longer sequence. This can be extended infinitely to the left giving the infinite exact sequence.

$$\cdots \to \widetilde{K}(S^2X) \to \widetilde{K}(S^2A) \to \widetilde{K}(SX/SA)) \to \widetilde{K}(SX) \to \widetilde{K}(SA) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A)$$

However, getting the sequence to extend infinitely to the right requires the following theorem.

Theorem 6.8. (Bott Periodicity) For any compact Hausdorff space X there is an isomorphism $\widetilde{K}(X) \cong \widetilde{K}(S^2X)$.

The proof of Bott periodicity is discussed in the following section. Applying Bott periodicity to the infinite exact sequence induced by the pair (X, A) immediately gives $\widetilde{K}(X) \cong \widetilde{K}(S^2X)$, $KR(A) \cong \widetilde{K}(S^2A)$. In fact, there are only 6 distinct group isomorphism classes in the infinite exact sequence. This justifies drawing the following cyclic commutative diagram. A cyclic exact sequence

$$\begin{array}{ccccc} \widetilde{K}(X/A) & \longrightarrow & \widetilde{K}(X) & \longrightarrow & \widetilde{K}(A) \\ \uparrow & & & & \downarrow \\ \widetilde{K}(SA) & \longleftarrow & \widetilde{K}(SX) & \longleftarrow & \widetilde{K}(SX/SA) \end{array}$$

FIGURE 5. Cyclic Exact Sequence

can be interpreted as an exact sequence extending infinitely in both direction by continuing around the cycle infinitely in both directions. However, before unraveling the cyclic sequence, adopt the following change in notation to negative indices. From now on, denote $\widetilde{K}(S^kX)$ by $\widetilde{K}^{-k}(X)$, denote $\widetilde{K}(S^kA)$ by $\widetilde{K}^{-k}(A)$ and denote $\widetilde{K}(S^kX/S^kA)$ by $\widetilde{K}^{-k}(X,A)$. However, by Bott periodicity and as depicted in Figure 5, there are 6 distinct group isomorphism classes:

$$\widetilde{K}^{-1}(X,A) \qquad \widetilde{K}^{-1}(X) \qquad \widetilde{K}^{-1}(A) \qquad \widetilde{K}^{0}(X,A) \qquad \widetilde{K}^{0}(X) \qquad \widetilde{K}^{0}(A)$$

This allows for the following definition for positive indices. Specifically, let $\widetilde{K}^k(A)$ be $\widetilde{K}^0(A)$ for even k and $\widetilde{K}^1(A)$ for odd k. Doing the same for $\widetilde{K}^k(X)$ and $\widetilde{K}^k(X,A)$ gives the following infinite sequence of groups.

$$\cdots \to \widetilde{K}^{-1}(X) \to \widetilde{K}^{-1}(A) \to \widetilde{K}^{0}(X,A) \to \widetilde{K}^{0}(X) \to \widetilde{K}^{0}(A) \to \widetilde{K}^{1}(X,A) \to \widetilde{K}^{1}(X) \to \cdots$$

This infinite sequence is indeed exact, for it is only a reformulation of the cyclic exact sequence shown in figure 5 and each functor $\widetilde{K}^k$ preserves homotopy by $\widetilde{K}$ preserves homotopy, thus this construction gives a valid cohomology theory.

4. Further Structure of K-theory

The external product is central to the proof of Bott periodicity and provides the structure necessary for the application in the following chapter.

Definition 6.9 (External Product)**.** Let X and Y be compact Hausdorff spaces and let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the natural projections. Then, the *external product* is a mapping $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ as defined by $\mu(a \otimes b) = p_1^*(a)p_2^*(b)$ and notated by $a * b$.

The external product of K-theory has particularly nice structure on spheres. This is due to considering the canonical line bundle H over $\mathbb{C}P^1$, and noting that by $S^2 \approx \mathbb{C}P^1$, the bundle H can also be viewed as a bundle over S^2. Further recall the relationship $(H \otimes H) \oplus \varepsilon^1 \approx H \oplus H$ as discussed in the Chapter 4. This can be denoted by $H^2 + 1 = 2H$, which can be more compactly written as $(H-1)^2 = 0$. Now, note $1, H \in K(S^2)$ and consider the subring $\mathbb{Z}[H]/(H-1)^2$ of $K(S^2)$ together with the inclusion $i : \mathbb{Z}[H]/(H-1)^2 \to K(S^2)$.

Claim 6.10. The mapping $i : \mathbb{Z}[H]/(H-1)^2 \to K(S^2)$ is an isomorphism.

The strategy of this proof is to consider the composition of the external product map μ and the above inclusion map i to get a mapping M.

$$M : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \xrightarrow{i} K(X) \otimes K(S^2) \xrightarrow{\mu} K(X \times S^2)$$

In fact, the above map is an isomorphism. Hatcher gives a full ten page proof of this in [**4**], but the strategy is to manipulate the form of the clutching functions to show injectivity and surjectivity. This result is more important than Claim 6.10 and so is now stated.

Theorem 6.11 (Fundamental Product Theorem)**.** $M : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times S^2)$ as described above is an isomorphism.

The Fundamental Product Theorem implies Claim 6.10 by taking X to be a point. This follows from $K(\{pt\}) = \mathbb{Z}$ together with $Z \otimes R \cong R$ for any ring R as discussed in Claim 2.11. Finally, note that substituting Claim 6.10 into Theorem 6.11 reveals that the map M is an isomorphism $K(X) \otimes K(S^2) \to K(X \times S^2)$, so in fact the mapping M is the external product μ, thus the

Fundamental Product Theorem is a statement about the external product. The takeaway is that the External product on K-theory has particularly useful structure on spheres.

Similar patterns on spheres arise in the external product of *reduced* K-theory.

Example 6.12. The fundamental product theorem gives that $K(S^2) \cong \mathbb{Z}[H]/(H-1)^2$, but the relationship between reduced K-theory and unreduced K-theory gives that $\widetilde{K}(S^2) \cong \mathbb{Z}$ with trivial multiplication. To see this, note that any element of $K(X)$ can be expressed by $k + mH$. However, this is better written as $n + m(H-1)$ so that n represents the dimension of the vector bundle and m can range without affecting the dimension. Now recall that $\widetilde{K}(S^2)$ is the kernel of the mapping $f^* : K(S^2) \to K(\{pt\})$ induced by $f : \{pt\} \to K(S^2)$. This pullback is exactly Example 4.17 with the domain space set to a point. Then $f^*(E)$ for any vector bundle E will give the trivial bundle of dimension E. In other words, $f^* : n + m(H-1) \mapsto n$. Any element any element $m(H-1)$ is then mapped to 0, so the kernel of f^* is the infinite cyclic group generated by $(H-1)$. However, the condition $(H-1)^2 = 0$ forces the product of any two elements in the kernel to be 0.

The external product on reduced K-theory is defined by taking the short exact sequence in Claim **??** with the initial space-subspace pair to be $X \wedge Y \subset X \times Y$. Considering the last three terms of the resulting sequence gives the following short exact sequence.

$$\widetilde{K}(X \wedge Y) \to \widetilde{K}(X \times Y) \to \widetilde{K}(X \vee Y)$$

As noted earlier, the splitting lemma always applies to this construction due to the relationship between inclusions and retracts. Thus it follows that $\widetilde{K}(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X \vee Y)$. Now note the following expressions for $K(X \wedge Y)$ and $K(X) \otimes K(Y)$, recalling the relationship $\widetilde{K}(X \vee Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y)$ discussed previously in Example 6.6 as well as the previously discussed properties of direct sum and tensor product.

$$\begin{aligned} K(X) \otimes K(Y) &\cong (\widetilde{K}(X) \oplus \widetilde{K}(\mathbb{Z})) \otimes (\widetilde{K}(Y) \oplus \widetilde{K}(\mathbb{Z})) \cong (\widetilde{K}(X) \otimes \widetilde{K}(Y)) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \\ K(X \times Y) &\cong (\widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X \vee Y)) \oplus \mathbb{Z} \cong (\widetilde{K}(X) \wedge \widetilde{K}(Y)) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \end{aligned}$$

Note the similarities between the expanded forms of $K(X) \otimes K(Y)$ and $K(X \times Y)$. Because nearly every term on the right hand side of the expanded form is equivalent, an the external product $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ will induce an external product of reduced K-theory $\widetilde{\mu} : \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X) \wedge \widetilde{K}(Y)$. This construction is discussed in more detail in [**4**].

This close relationship between μ and $\widetilde{\mu}$, the Fundamental Product Theorem gifts reduced K-theory with pleasant properties. For instance, μ is an isomorphism when taking $Y = S^2$ by the Fundamental Product Theorem. But then this forces $\widetilde{\mu}$ to be an isomorphism and thus $\widetilde{K}(X) \otimes \widetilde{K}(S^2) \cong \widetilde{K}(X) \wedge \widetilde{K}(S^2)$. Due to the quotient map $S^n \to S^n \wedge X$ discussed in Example 3.28 in combination with Claim 6.7, this gives an isomorphism $\widetilde{K}(X) \wedge \widetilde{K}(S^2) \cong \widetilde{K}(S^2 X)$. Additionally, by Example 6.7, $\widetilde{K}(S^2) \cong \mathbb{Z}$ and thus $\widetilde{K}(X) \otimes \widetilde{K}(S^2) \cong \widetilde{K}(X)$. Combining all of these isomorphisms gives the following result.

$$\widetilde{K}(X) \cong \widetilde{K}(X) \otimes \widetilde{K}(S^2) \cong \widetilde{K}(X) \wedge \widetilde{K}(S^2) \cong \widetilde{K}(S^2 X)$$

Taking X to be S^0 and iteratively applying the above isomorphism provides the result $\widetilde{K}(S^0) \cong \widetilde{K}(S^{2n})$ for any n; similarly, taking X to be S^1 gives $\widetilde{K}(S^1) \cong \widetilde{K}(S^{2n+1})$, which justifies the Bott Periodicity Theorem stated previously as Theorem 6.8.

Structure on the $\widetilde{K}$ functor is learned from the functor K, but now $\widetilde{K}$ returns the favor and gives insight to $K(S^{2k})$.

Example 6.13. Consider $K(S^{2k})$ for some K. By Claim 6.4 and Bott Periodicity,

$$K(S^{2k}) \cong \widetilde{K}(S^{2k}) \oplus \mathbb{Z} \cong \widetilde{K}(S^2) \oplus \mathbb{Z} \cong K(S^2)$$

Thus $\widetilde{K}(S^{2k}) \oplus \mathbb{Z} \cong \mathbb{Z}[a]/(a-1)^2$, which is equivalent to $\widetilde{K}(S^{2k}) \oplus \mathbb{Z} \cong \mathbb{Z}[\gamma]/(\gamma)^2$ by a change of variables.

Example 6.14. Now consider $K(S^{2k} \times S^{2l})$. By sequentially applying the External Product Theorem, the result from Example 6.13 and the result from Example 2.10, it follows that

$$K(S^{2k} \times S^{2l}) \cong K(S^{2k}) \otimes K(S^{2l}) \cong \mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta]/(\beta^2) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$$

Additionally, by tracing the definition of the external product through the isomorphism chain, it follows that $\alpha = p_1^*(\gamma)$ and $\beta = p_2^*(\gamma)$ if p_1 is the projection $S^{2k} \times S^{2l} \to S^{2k}$ and p_2 is the projection $S^{2k} \times S^{2l} \to S^{2l}$.

CHAPTER 7

Division Algebra Application

Recall from Definition 1.7 that a division algebra is a ring with multiplicative inverses. Further recall that a division algebra structure in $\mathbb{R}^n$ induces an H-space structure over the sphere S^{n-1} by considering the subset of $\mathbb{R}^n$ with norm 1. And so if a division algebra structure on $\mathbb{R}^n$ exists, then there is an H-space structure on the sphere S^{n-1}. As Bott periodicity hints at, K-theory has a close relationship with spheres, making K-theory a good tool to examine the existence of H-spaces on spheres. This chapter will use K-theory to show that an H-space structure cannot exist on any sphere other than S^0, S^1, S^3, and S^7. This conclusion regarding H-spaces together with the explicit examples of the reals, the complex numbers, the quaternions, and the octonions gives the following theorem.

Theorem 7.1. $\mathbb{R}^n$ is a division algebra only when n is 1, 2, 4, or 8.

1. The odd case

The argument will first rule out the possibility of a division algebra structure on odd dimensions other than 1, so assume for the purpose of contradiction that there is a division algebra structure on $\mathbb{R}^{2k+1}$ for some positive integer k. It then follows that S^{2k} is an H-Space, which promises a continuous mapping $\mu : S^{2k} \times S^{2k} \to S^{2k}$. Additionally let p_1 denote the projection from $S^{2k} \times S^{2k}$ to the first factor and let p_2 be the projection to the second factor. Applying the K-theory functor to the H-space multiplication mapping gives a homomorphism between rings $\mu^* : K(S^{2k}) \to K(S^{2k} \times S^{2k})$. By Example 6.14, the homomorphism is of the form $\mu^* : \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ such that $\alpha = p_1^*(\gamma)$ and $\beta = p_2^*(\gamma)$. Note that γ is the generator of the ring and in particular, γ is $H - 1$ where H denotes the canonical line bundle over the space.

The contradiction will arise in analyzing the quantity $\mu^*(\gamma)$. To accomplish this, define $i_1 : S^{2k} \to S^{2k} \times S^{2k}$ by $i_1 : x \mapsto (x, e)$ where e is the identity element of S^{2k} as an H-space. Note that $\mu \circ i_1$ is the identity, giving $i_1^* \circ \mu^*$ is the identity, so studying i_1^* will give information about μ^*.

It follows from the definition of i_1 that $p_1 \circ i_1 = \mathrm{Id}$ and so $i_1^* \circ p_1^*$ is identity. Plugging in α to both sides and recalling the definition of α then gives:

$$i_1^*(\alpha) = \gamma$$

In a similar way, it follows that $p_2 \circ i_1$ is a constant function always mapping to the H-space identity point e. Denote this by $p_2 \circ i_1 = \mathrm{const}_e$, which gives $i_1^* \circ p_2^* = \mathrm{const}_e^*$. Again plug in γ to both sides and recall the definition of β and thus $i_1^*(\beta) = \mathrm{const}_e^*(\gamma)$.

To simplify this further, recall that γ is $H - 1$ where H is the canonical line bundle. And note that because each fiber of H is of dimension 1 and const_e maps to a point, the pullback $\mathrm{const}_e^*(H)$ is the trivial bundle ε^1, which is the multiplicative identity denoted by 1. Thus by the ring homomorphism properties:

$$\text{const}^*(\gamma) = \text{const}^*(H-1) = \text{const}^*(H) - \text{const}^*(1) = 1 - 1 = 0$$

This gives the following crucial piece of information.

$$i_1^*(\beta) = 0$$

Now return to analyzing the quantity $\mu^*(\gamma)$. As an element of $\mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2)$, the quantity is of the form $\mu^*(\gamma) = n + a\alpha + b\beta + m\alpha\beta$ for some integers a, b, n, m. However, now apply $i_1^* \circ \mu^* = \text{Id}$ with the information $i_1^*(\alpha) = \gamma$ and $i_1^*(\beta) = 0$ and keeping ring homomorphism properties in mind:

$$\gamma = i_1^*(\mu^*(\gamma)) = i_1^*(n + a\alpha + b\beta + m\alpha\beta) = n + a \cdot i_1^*(\alpha) + b \cdot i_1^*(\beta) + m \cdot i_1^*(\beta) \cdot i_1^*(\alpha) = n + a\gamma$$

And thus by $\gamma = n + a\gamma$, it follows that $n = 0$ and $a = 1$. Applying the same argument by considering the inclusion $i_2 : S^{2k} \to S^{2k} \times S^{2k}$ by $i_2 : x \mapsto (e, x)$ will give that $b = 1$. And so μ^* can be written in the reduced form

$$\mu^* = \alpha + \beta + m\alpha\beta$$

The contradiction arises from the observation that the relation $\gamma^2 = 0$ gives that $(\mu^*(\gamma))^2 = 0$. However, the derived expression for $\mu^*(\gamma)$ and the relations $\alpha^2 = \beta^2 = 0$ imply a different result.

$$(\mu^*(\gamma))^2 = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta \neq 0$$

The above proof is an elaboration of the proof given in [**4**].

2. The even case

This section will closely follow the discussion in [**4**], showing that if a division algebra structure exists on $\mathbb{R}^{2n}$, then n must be 1, 2, or 4. To begin, however, assume a division algebra structure on $\mathbb{R}^k$ without any restrictions on k. This induces an H-Space multiplication $\mu : S^{k-1} \times S^{k-1} \to S^{k-1}$. The goal is to use this multiplication to construct a map $\hat{\mu} : S^{2k-1} \to S^k$. Before the general construction, first consider the particular case of $k = 0$, and let $S^0 = \{0, 1\}$. Now let $\mu : S^0 \times S^0 \to S^0$ be given by $\mu : (x, y) \mapsto x$. Now represent the domain S^1 of $\hat{\mu}$ by the boundary $\partial(D^1 \times D^1)$, which is equivalent to $\partial(D^1) \times D^1 \cup D^1 \times \partial(D^1)$, which is again equivalent to $S^0 \times I \cup I \times S^0$. Represent the codomain S^1 of $\hat{\mu}$ as the union of the two hemispheres, $D_1^+ \cup D_1^-$ with their boundaries S^0 associated. Now let the map $\hat{\mu} : S^1 \to S^1$ be defined by taking these representations $\hat{\mu} : S^0 \times I \cup I \times S^0 \to D_1^+ \cup D_1^-$ as follows

$$\hat{\mu}(x, y) = \begin{cases} y \cdot g(x, 1) \in D_+^1 & \text{if } (x, y) \in I \times S^0 \\ x \cdot g(1, y) \in D_-^1 & \text{if } (x, y) \in S^0 \times I \end{cases}$$

The general construction takes the map $\mu : S^{k-1} \times S^{k-1} \to S^{k-1}$ to a new map map $\hat{\mu} : S^{2k-1} \to S^k$ by expressing the domain and codomain in the map by $\hat{\mu} : \partial(D^k) \times D^k \cup D^k \times \partial(D^k)$ together with the following definition.

$$\hat{\mu}(x, y) = \begin{cases} |y| \cdot g(x, y/|y|) \in D_+^k & \text{if } (x, y) \in D^k \times \partial(D^k) \\ |x| \cdot g(x/|x|, y) \in D_-^k & \text{if } (x, y) \in \partial(D^k) \times D^k \end{cases}$$

Now place the restriction that k is even, letting $k = 2n$. This provides a mapping $\hat{\mu} : S^{4n-1} \to S^{2n}$. This mapping allows for the disk D^{4n} to be attached to S^{2n}, resulting in a combination C_f. Specifically, let $C_f = S^{2n} \cup^* D^{4n}/\sim$ for an equivalence relation $\sim$ defined by $x \sim y$ for $x \in D^{4n}$

and $y \in S^{2n}$ if $\hat{\mu}(x) = y$. Additionally note that $C_f/S^{2n} \approx S^{4n}$ and thus taking the space-subspace pair (C_f, S^{2n}) together with the quotient and inclusion maps induces the following exact sequence.

$$\widetilde{K}(S^{4n}) \xrightarrow{i^*} \widetilde{K}(C_f) \xrightarrow{q^*} \widetilde{K}(S^{2n})$$

Now let $\alpha \in \widetilde{K}(C_f)$ be the image of the generator of the cyclic group $\widetilde{K}(S^{4n})$ under the map i^* and let $\beta \in \widetilde{K}(C_f)$ map to the generator of the cyclic group $\widetilde{K}(S^{4n})$ through the map q^*. Note $q^*(\beta)^2 = q^*(\beta)$, which is simply 0 by the multiplication in $\widetilde{K}(S^{2n})$ trivial and thus $\beta^2 \in \mathrm{Ker}(q^*)$. But by the sequence exact, $\beta^2 \in \mathrm{Im}(i^*)$ and the image is generated by the element α. Thus $\beta^2 = h\alpha$ for some integer h. This integer h is the *Hopf invariant* of μ, and is key to the proof. It follows from the H-Space properties of μ that h must be ± 1, which is fully described in [**4**].

The *Adams operations* is a useful family of homomorphisms between K-theory rings. In this case, Adams operations demonstrate that for a map $\hat{\mu} : S^{4n-1} \to S^{2n}$ to have Hopf invariant ± 1, it must be that $n = 1$, 2, or 4 which will give that $\mathbb{R}^{2n}$ can only be of dimension 2, 4, or 8. In particular, the Adams operations is a family of ring homomorphisms $\psi^k : K(X) \to K(X)$ indexed over the nonnegative integers that satisfies the following pleasant properties.

(i) $\psi^k f^* = f^* \psi^k$.
(ii) $\psi^k(L) = L^k$ for any line bundle L.
(iii) $\psi^k \circ \psi^l = \psi^{kl}$ for all k and l.
(iv) $\psi^p(\alpha) \equiv \alpha^p \pmod p$ for any prime p.

The construction of such a set of operations to demonstrate existence is technical and a full proof is given in [**4**, p. 62-64].

Accepting the existence of such a family of ring homomorphisms, examine how these operations act on base space $\widetilde{K}(S^{2n})$. First consider the $n = 1$ case, letting $\alpha = H - 1$ where H is the canonical line bundle over $\mathbb{C}P^1$. Both H and 1 are line bundles, property (ii) of the Adams operations with the homomorphism properties gives the following result.

$$\psi^k(\alpha) = \psi^k(H - 1) = \psi^k(H) - \psi^k(1) = H^k - 1 = (\alpha + 1)^k - 1$$

Because the multiplication in $\widetilde{K}(S^2)$ is trivial, $\alpha^2 = 0$. Thus in the expansion of $(\alpha + 1)^k$, only the 0 and 1 degree terms will survive, which gives the following simplification.

$$\psi^k(\alpha) = (\alpha + 1)^k - 1 = k\alpha + 1 - 1 = k\alpha$$

This result on the generator of $\widetilde{K}(S^2)$ gives that the Adams operations ψ^k act by $\psi^k(E) = kE$ for any $E \in \widetilde{K}(S^2)$. An induction argument using the external product gives the more general conclusion that $\psi^k(E) = k^n E$ over $\widetilde{K}(S^{2n})$.

Now return to the map μ and the induced variables α and β representing the image of the $\widetilde{K}(S^{4n})$ generator and the preimage of the $\widetilde{K}(S^{2n})$ of the exact sequence respectively. Then, α is the generator of the subring $\mathrm{Im}(i^*) \subset \widetilde{K}(C_f)$, which is isomorphic to $\widetilde{K}(S^{4n})$ and thus the result discussed in the previous paragraph applies and gives $\psi^k(\alpha) = k^{2n}\alpha$. Similarly, β is related to the generator of a subgroup of $\widetilde{K}(C_f)$ isomorphic to $\widetilde{K}(S^{2n})$. A difference, however, is that β may be offset from the generator of this subring by some element of the kernel. Thus we can only write $\psi^k(\beta) = k^n\beta + \mu_k\alpha$ for some $\mu_k \in \mathbb{Z}$.

This expression of $\psi^k(\beta)$ immediately gives some useful information. Combining the expression $\beta^2 = h\alpha$ for Hopf invariant α with $\psi^2(\beta) \equiv \beta^2 \pmod 2$ as promised by Property (iv) gives

$\psi^2(\beta) \equiv h\alpha \pmod 2$. Now incorporate the expression for $\psi^2(\beta)$ derived in the previous paragraph, which gives $2^n\beta + \mu_k\alpha \equiv h\alpha \pmod 2$. This expression then gives $\mu_2 \equiv h \pmod 2$ and so μ_2 is odd by $h = \pm 1$, which will be of use later.

Now use the expressions for $\psi^k(\alpha)$ and $\psi^k(\beta)$ to expand the expression $\psi^2\psi^3(\beta)$.

$$\psi^2\psi^3(\beta) = \psi^2(3^n\beta + \mu_3\alpha) = 2^n3^n\beta + (2^{2n}\mu_3 + 2^n\mu_3)\alpha$$

Note the same the commutativity of the Adams operations by $\psi^2\psi^3 = \psi^6 = \psi^3\psi^2$. Thus repeating the same expansion for $\psi^3\psi^2(\beta)$ gives the relationship

$$2^n3^n\beta + (2^{2n}\mu_3 + 3^n\mu_2)\alpha = 3^n2^n\beta + (3^{2n}\mu_2 + 2^n\mu_3)\alpha$$

Canceling the β term from each side and canceling out the remaining α, and rearranging gives the following result.

$$2^n(2^n - 1)\mu_3 = 3^n(3^n - 1)\mu_2$$

And thus 2^n divides the right hand side. By μ_2 odd, as noted earlier, it must be that 2^n divides $3^n - 1$. An elememtary number theory proof given in [**4**] gives that 2^n only divides $3^n - 1$ when n is 1, 2, or 4.

Bibliography

[1] M. Atiyah. *K-theory*. CRC Press, 2018.
[2] D.S. Dummit and R.M. Foote. *Abstract Algebra*. Wiley, 2004.
[3] A Hatcher. Algebraic topology, cambridge univ. *Press, Cambridge xii*, 2002.
[4] A. Hatcher. *Vector Bundles and K-Theory*. 2003. http://www.math.cornell.edu/~hatcher.