General Exam Paper

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January 14, 2025

1 Introduction

Consider a closed Riemannian manifold M with set $\mathcal{G}$ of unit-speed closed geodesics. The *X-ray transform* $I_0 : C^\infty(M) \to \ell^\infty(\mathcal{G})$ is given by integrating a given smooth function f over every closed geodesic $\gamma \in \mathcal{G}$:

$$(I_0 f)(\gamma) \coloneqq \int_0^{\ell(\gamma)} f(\gamma(t))dt$$

where $\ell(\gamma)$ denotes the length of γ. A natural question is the injectivity of this X-ray transform: do the integral values over all closed geodesics uniquely identify the function? We will see the answer depends on the dynamics of the geodesic flow over the manifold, which in turn depends on the geometry of the manifold M.

As a first example, we consider the case of the X-ray transform over the sphere $\mathbb{S}^2$, which was originally studied by Funk [Fun13] and is often called the *Funk Transform.* In this case, the set of closed geodesics $\mathcal{G}$ over $\mathbb{S}^2$ are all the great circles. Is the X-ray transform over $\mathbb{S}^2$ injective? Note any function $f \in C^\infty(\mathbb{S}^2)$ that is odd, meaning $f(-x) = -f(x)$, will satisfy $I_0 f = 0$. As there are many nonzero odd functions, this demonstrates the X-ray transform is not injective over the sphere.

Next consider the example of a closed hyperbolic surface $\mathbb{H}/\Gamma$ where $\mathbb{H}$ is the hyperbolic half-plane of constant Gaussian curvature -1, and Γ is a discrete group acting freely and properly on $\mathbb{H}$. In this case, the collection $\mathcal{G}$ of closed geodesics on $\mathbb{H}/\Gamma$ is significantly more complex and chaotic, but $\mathcal{G}$ will be countable with and contain a unique closed geodesic in each free homotopy class. It turns out the X-ray transform over $\mathbb{H}/\Gamma$ is injective and much of this document is dedicated to partially explaining why. A key distinction between the case of $\mathbb{S}^2$ and the case of $\mathbb{H}/\Gamma$ is that geodesics over negatively curved manifolds have more chaotic behavior than geodesics over positively curved manifolds. We will see this key property is that $\mathbb{H}/\Gamma$ has an *Anosov* metric, meaning the geodesic flow is sufficiently chaotic and will be precisely defined in Section 2.2, and in fact I_0 is injective for all Anosov manifolds.

The question of injectivity of the X-ray transform over closed Anosov manifolds has a few notable applications and historic motivations. First, given such a closed Anosov manifold M, this question is closely related to the problem of *marked length rigidity*; that is, to what extend does the *length spectrum* L_g, which encodes the lengths of all closed geodesics, determine the underlying Riemannian metric g? The *Burns-Katok* conjecture [BK85] is that for any two Anosov metrics g_1, g_2 giving the same length spectrum $L_{g_1} = L_{g_2}$, then there exists a diffeomorphism ϕ isometric to the identity such that $\phi^* g_1 = g_2$. This conjecture was first proven true for conformal metrics [Kat88], then for negatively curved surfaces [Cro90, Ota90], when one of the metrics is locally symmetric and the other is negatively curved [Ham99], then for negatively curved metrics close together [GL19], and most recently for Anosov surfaces [GLP23]. Importantly, the local problem of *infinitesimal marked length rigidity* – if a small perturbation of an Anosov metric is given by the pullback of such an isotopy – is equivalent to the injectivity of an X-ray transform I_2 on symmetric 2-tensors. While we will discuss the injectivity of the X-ray transform I_0 on functions for simplicity, much of the theory caries over to I_2.

In fact, the injectivity of the X-ray transform is related to Kac's famous question "can one hear the shape of a drum?" [Kac66], which is the question of *spectral rigidity*: does the Laplace spectrum of a manifold determine the underlying Riemannian metric up to isometry? It was conjectured that perhaps the answer is positive for Anosov manifolds, but Vigneras [Vig80] found a pair of isospectral hyperbolic surfaces that are not isometric. However, there is still the question of *infinitesimal spectral rigidity* on Anosov manifolds: must any isospectral family of perturbations of a metric be given by the pullbacks of an isotopy? The answer is yes [GK80b, CS98, GL19]: infinitesimal spectral rigidity is implied by infinitesimal marked length rigidity, which is in turn implied by the injectivity of I_2, which is proven by similar techniques to the case of I_0, which hopefully provides sufficient reason to learn some of these techniques in this document.

While this document will focus on the injectivity of the X-ray transform over Anosov manifolds, there is a related X-ray transform on manifolds with boundary in which the X-ray data is given by the integrals of an unknown function over all geodesics that begin and terminate at the boundary. In this case, the X-ray transform is injective so long as the manifold M with boundary is *simple*, meaning M is simply connected and has no conjugate points. The study of the X-ray transform on simple manifolds and the X-ray transform on Anosov manifolds are often in analogy and structurally similar. For example, the question of injectivity of the simple manifold X-ray transform on 2-tensors follows naturally from the widely studied *boundary rigidity* inverse problem in a similar way the question of injectivity of I_2 on Anosov manifolds follows from the question of marked length rigidity. In fact, it was recently proven in [EL24] that spectral rigidity for Anosov manifolds implies boundary rigidity for certain manifolds (including simple manifolds) by using [CEG23] to embed these manifolds with boundary into Anosov manifolds.

2 Preliminaries

2.1 Unit tangent bundle and geodesic flow

For any Riemannian manifold (M, g), the *unit tangent bundle* $SM = \{(x, v) \in TM : |v|_g = 1\}$ is all unit-length elements of the tangent bundle, which inherits the natural projection $\pi_0 : SM \to M$. The unit tangent bundle is the phase space for the *geodesic flow* $\varphi_t : SM \to SM$ given by $\varphi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$ where $\gamma_{x,v}(t)$ denotes the unique geodesic with initial position $\gamma_{x,v}(0) = x$ and initial velocity $\dot{\gamma}_{x,v}(0) = v$. We denote by X the smooth vector field over SM generating this geodesic flow. The dynamics of this geodesic flow plays a crucial role in studying the X-ray transform, so we discuss the geometry of its phase space SM.

The unit tangent bundle SM inherits a natural metric from M, which we now describe. Note smooth curves $Z : (-\varepsilon, \varepsilon) \to SM$ over the unit tangent bundle take the form $Z(t) = (\alpha(t), W(t))$ for a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to M$ over the base manifold together with a smooth vector field $W(t) \in S_{\alpha(t)}M$ over $\alpha(t)$. Thus we may think of a tangent vector $\xi \in T_{(x,v)}SM$ as the velocity vector of such a curve $\xi = \frac{d}{dt}\big|_{t=0} Z(t)$ so that $\alpha(0) = x$ and $W(0) = v$. Now note the Levi-Civita connection ∇ and corresponding covariant derivative D_t along $\alpha(t)$ allow for the natural identification

$$\xi = \frac{d}{dt}\bigg|_{t=0} (\alpha(t), W(t)) \leftrightarrow \left(\frac{d}{dt}\bigg|_{t=0} \alpha(t),\, D_t W(t)|_{t=0} \right) = (d\pi_0 \xi, \mathtt{K}\xi)$$

where the *connection map* $\mathtt{K} : TSM \to TM$ is given by $\mathtt{K}\xi \coloneqq D_t W|_{t=0}$. Then the metric g on M naturally induces the *Sasaki metric* G on SM by

$$\langle \xi, \eta \rangle_G \coloneqq \langle d\pi_0 \xi, d\pi_0 \eta \rangle_g + \langle \mathtt{K}\xi, \mathtt{K}\eta \rangle_g$$

for $\xi, \eta \in T_{(x,v)}SM$. In fact, defining the *vertical bundle* $\mathbb{V} \coloneqq \ker d\pi_0$ and the bundle $\widetilde{\mathbb{H}} \coloneqq \ker \mathtt{K}$ we have a splitting $TSM = \mathbb{V} \oplus \widetilde{\mathbb{H}}$ and linear isomorphisms $d\pi_0 : \widetilde{\mathbb{H}} \to TM$ and $\mathtt{K} : \mathbb{V} \to TM$ onto their images. The Sasaki metric is then defined by declaring these isomorphisms to be isometries and declaring the splitting $TSM = \mathbb{V} \oplus \widetilde{\mathbb{H}}$ to be orthogonal. Additionally, note the definition of the Levi-Civita connection ensures the generator X of the geodesic flow satisfies $X \in \ker \mathtt{K} = \widetilde{\mathbb{H}}$ and so we make the orthogonal decomposition $\widetilde{\mathbb{H}} = \mathbb{X} \oplus \mathbb{H}$ into the *flow direction* $\mathbb{X} \coloneqq \mathbb{R}X$ and the *horizontal bundle* $\mathbb{H}$ and so we have the orthogonal decomposition

$$TSM = \mathbb{X} \oplus \mathbb{H} \oplus \mathbb{V}.$$

In the case M is a surface, the subbundles $\mathbb{X}$, $\mathbb{H}$, and $\mathbb{V}$ are rank 1 and spanned by the unit-length smooth vector fields X, H, V respectively. By local coordinate computations, we may compute the Lie algebra of this frame, often called the *structure equations*:

$$[X, V] = H, \qquad [H, V] = -X, \qquad [X, H] = -KV. \tag{1}$$

where K denotes the Gaussian curvature function. Note that by Liouville's theorem we have $\operatorname{div} X = 0$. We also have $\operatorname{div} H = \operatorname{div} V = 0$ and so we have the integration by parts formulas

$$\begin{aligned}(Xu, w)_{L^2(SM)} &= -(u, Xw)_{L^2(SM)}\\ (Hu, w)_{L^2(SM)} &= -(u, Hw)_{L^2(SM)}\\ (Vu, w)_{L^2(SM)} &= -(u, Vw)_{L^2(SM)}.\end{aligned}$$

See [PSU23] for more details.

2.2 Anosov Flows and Manifolds

The reason the X-ray transform on $\mathbb{S}^2$ is not injective while the X-ray transform on a hyperbolic surface $\mathbb{D}/\Gamma$ is injective is because the geodesic flow over surfaces of negative curvature has chaotic properties such as ergodicity. In fact, Anosov [Ano69] first proved the geodesic flow over manifolds with negative sectional curvature is ergodic for arbitrary dimension by showing the flow has a key property, which is now referred to as the Anosov property. This Anosov property is the key property that allows for proving injectivity of the X-ray transform via microlocal techniques. Consider a compact manifold $\mathcal{M}$ with flow φ_t and infinitesimal generator X. The flow φ_t is *Anosov* if there is a continuous and flow-invariant splitting of the tangent space

$$T\mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u$$

into the flow direction, the *stable bundle*, and the *unstable bundle* respectively such that given an arbitrary metric $|\cdot|$ on $\mathcal{M}$ there exists constants $C, \lambda > 0$ so that for all $t \geq 0$

$$\begin{aligned}|d\varphi_t(v)| &\leq Ce^{-\lambda t}|v| \quad \text{for } v \in E_s,\\ |d\varphi_{-t}(v)| &\leq Ce^{-\lambda t}|v| \quad \text{for } v \in E_u.\end{aligned}$$

That is, Anosov flows have a strong sensitivity to initial conditions at every point – a small perturbation in a non-flow direction will result in a drastically different trajectories in either the forward or backward flow direction. To apply the microlocal tools of Section 2.3, we are interested in the dual splitting

$$T^*\mathcal{M} = E_0^* \oplus E_s^* \oplus E_u^* \tag{2}$$

defined so that $E_0^*(E_s \oplus E_u) = 0$, $E_s^*(E_s \oplus \mathbb{R}X) = 0$, and $E_u^*(E_u \oplus \mathbb{R}X) = 0$. For all Anosov flows, the collection of points belonging to periodic orbits is dense in $\mathcal{M}$. Often times, flows will preserve a volume form μ (for instance, any Hamiltonian system preserves the Liouville volume form) in which case the flow is called *volume preserving*. Any volume preserving Anosov flow is *topologically transitive*, meaning there exists a point with dense orbit, and *ergodic*, meaning $\ker X|_{L^2_\mu}$ contains only the constants. We are interested in the geodesic flow

$\varphi_t : SM \to SM$ over a Riemannian manifold M, which is a volume preserving flow by taking the *Liouville form* $d\Sigma$, defined as the volume form induced by the Sasaki metric on SM. If this flow is Anosov, we say M is an *Anosov manifold*, which includes all manifolds with strictly negative sectional curvature. In this case, there is a unique periodic orbit in each free homotopy class of M. For proofs of the above claims, see [Lef25, Chapter 8].

2.3 Microlocal tools

In the past fifteen years, techniques from microlocal analysis have been applied to the study of the dynamics of Anosov flows with much success, including results on the injectivity of the X-ray transform. Due to the dual splitting (2), it is often required to treat different direction of the cotangent bundle of the phase space differently, and microlocal analysis is precisely the tool for this. We review some of the relevant notions from microlocal analysis here.

We will always work over a closed Riemannian manifold M with metric g. The space of *distributions* $\mathcal{D}'(M)$ over M is the dual space of continuous linear functionals $C^\infty(M) \to \mathbb{C}$ and is equipped with the weak-star topology of the standard seminorm topology on $C^\infty(M)$. Recall a *pseudo-differential operator of order* m over an open subset $X \subset \mathbb{R}^n$ is a linear operator $A : C_c^\infty(X) \to C^\infty(X)$ defined using the Fourier transform by

$$Af = \mathrm{Op}(a)f \coloneqq (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\widehat{f}(x,\xi)d\xi \tag{3}$$

where $a(x,\xi) \in C^\infty(T^*X)$ is a *symbol of order* m, meaning for some $\rho, \delta \in [0,1]$ it satisfies the bound

$$|\partial_\xi^\beta \partial_x^\alpha a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|}. \tag{4}$$

We denote by $S^m_{\rho,\delta}(T^*X)$ the class of all symbols satisfying (4) and we write $\Psi^m_{\rho,\delta}(X) = \{\mathrm{Op}(a) : a \in S^m_{\rho,\delta}(T^*X)\}$ to denote the corresponding class of pseudodifferential operators. It is most common to take $\rho = 1$ and $\delta = 0$, but we will need more general ρ and δ. Note, for example, every differential operator of order m is a pseudodifferential operator of order m for any $\rho, \delta \in [0,1]$. Each pseudodifferential operator $A = \mathrm{Op}(a)$ has a well-defined *full symbol* $\sigma_{\mathrm{full}}(A) = a$, but unfortunately this full symbol is not invariant under a change of coordinates. However, the leading part of this full symbol is called the *principal symbol* which is well-defined on the cotangent bundle and is defined as the equivalence class under addition

$$\sigma_{\mathrm{pr}}(A) \in S^m_{\rho,\delta}(T^*X)/S^{m-(2\rho-1)}_{\rho,\delta}(T^*X). \tag{5}$$

Over a closed Riemannian manifold M of dimension n, we say a linear operator $A : C^\infty(M) \to C^\infty(M)$ is a pseudodifferential operator of order m if for any coordinate chart $\kappa : U \to V$ for $U \subset M$, $V \subset \mathbb{R}^n$ and bump functions $\phi, \psi \in$

$C_c^\infty(U)$ identically 1 on $U' \subset U$, we have that

$$(\kappa^{-1})^* \psi A \phi \kappa^* \in \Psi^m_{\rho,\delta}(\mathbb{R}^n). \tag{6}$$

where ψ and ϕ are multiplication operations in the above composition. We denote by $\Psi^m_{\rho,\delta}(M)$ the space of pseudodifferential operators over M, which again includes differential operators of order m over M. We may locally define the (full) symbol $\sigma^\kappa_{\text{full}}(A)$ of a pseudodifferential operator A by taking the pullback of the symbol of (6) although this depends on the coordinate chart κ. However, we may also locally define the principal symbol $\sigma^\kappa_{\text{pr}}(A)$ by taking the pullback of the principal symbol of (6), which is well-defined on the cotangent bundle T^*U'. Then we may defined a global principal symbol well-defined on T^*M by taking an atlas $\kappa_i : U_i \to V_i$ and partition of unity χ_i subordinate to $U_i' \subset U_i$, and defining

$$\sigma_{\text{pr}}(A) \coloneqq \sum \chi_i \sigma^{\kappa_i}_{\text{pr}}(A).$$

A pseudodifferential operator A of order m over a closed manifold M is called *elliptic* if there exists some $C, M > 0$ so that its principal symbol $\sigma_{\text{pr}}(A) = a(x,\xi)$ satisfies $a(x,\xi) \geq |\xi|^m/M$ for all $|\xi| > C$. Fix an elliptic pseudodifferential operator $\Lambda_s \in \Psi^s_{1,0}(M)$ for $s \in \mathbb{R}$. Then the *Sobolev space of order s* is given by $H^s(M) \coloneqq \Lambda_s^{-1}(L^2(M)) \subset D'(M)$ which comes with inner product $\langle u, v\rangle_{H^s} \coloneqq \langle \Lambda_s u, \Lambda_s v\rangle_{L^2}$. These Sobolev spaces are designed so that any pseudodifferential operator A of order m extends to a bounded operator $A : H^s(M) \to H^{s-m}(M)$.

We may alter the above theory to specify anisotropic pseudodifferential operators and anisotropic Sobolev spaces, which have variable orders in different directions. To specify this order, we take an *order function* $m(x,\xi) \in S^0(T^*M)$ that is homogeneous of order 0 in ξ for $|\xi| > 1$. Then we may define *anisotropic pseudodifferential operators* of order $m(x,\xi)$ in the same way we defined pseudodifferential operators above, but replacing the constant m with the variable $m(x,\xi)$. By taking an elliptic pseudodifferential operator $\Lambda_{m(\cdot)}$ of variable order $m(x,\xi)$, we may define *anisotropic Sobolev spaces* as $H^{m(\cdot)}(M) \coloneqq \Lambda^{-1}_{m(\cdot)}(L^2(M)) \subset D'(M)$ which comes with natural inner product $\langle u, v\rangle_{H^{m(\cdot)}} \coloneqq \left\langle \Lambda_{m(\cdot)} u, \Lambda_{m(\cdot)} v\right\rangle_{L^2}$.

Finally, we review the definition of the wave front set. Recall that for $u \in \mathcal{E}'(\mathbb{R}^n)$, a distribution of compact support, then u is smooth exactly when its Fourier transform $\widehat{u}$ is rapidly decreasing (i.e. a Schwartz function). That is, u fails to be smooth because there are particular directions in frequency space in which $\widehat{u}$ fails to be rapidly decreasing. It is helpful to identify these directions, and this is the idea of the wave front set. In fact, we may define the wave front set on $\mathbb{R}^n$ using exactly this idea, then lift this to a manifold, but we choose to write an equivalent coordinate-free definition using pseudodifferential operators. Indeed, we define the *wave front set* of a distribution $u \in D'(M)$ on a closed Riemannian manifold M to be

$$\text{WF}(u) = \{(x,\xi) \in T^*M : \forall A \in \Psi^0_{1,0}(M), Au \in C^\infty(M) \implies \sigma_{\text{pr}}(A)(x,\xi) = 0\}.$$

Importantly, if $\text{WF}(u) = \varnothing$, then $u \in C^\infty(M)$.

3 The X-ray transform on Anosov manifolds

Consider a closed manifold $\mathcal{M}$ with Anosov flow generated by X, which will have countable set $\mathcal{G}$ of closed orbits. Then we may generally define the *X-ray* transform $I : C^0(\mathcal{M}) \to \ell^\infty(\mathcal{G})$ so that for $f \in C^0(\mathcal{M})$ and $\gamma \in \mathcal{G}$ we set

$$If(\gamma) := \int_0^{\ell(\gamma)} f(\varphi_t(x))dt \tag{7}$$

where $\ell(\gamma)$ is the period of γ and $x \in \gamma$ is arbitrary. Then in the case of an Anosov manifold M, we may write the X-ray transform $I_0 : C^0(M) \to \ell^\infty(\mathcal{G})$ defined in Section 1 as $I_0 = I \circ \pi_0^*$. As recorded in the following theorem, this operator is injective, and this document will largely focus on the techniques in showing this injectivity.

Theorem 3.1. *I_0 is injective over Anosov manifolds of dimension at least 2.*

Guillemin and Kazhdan [GK80b] showed injectivity for surfaces with strictly negative curvature, Croke and Sharafutdinov [CS98] generalized this to higher dimensional manifolds with non-positive sectional curvature, then Dairbekov and Sharafutdinov [DS03] obtained injectivity of I_0 for Anosov manifolds in general. Below we will outline a proof of Theorem 3.1 for surfaces of strictly negative curvature.

The strategy for each of these injectivity results is to study solutions to the *transport equation* $Xu = f$ where X is the generator of the geodesic flow over SM. In particular, the strategy has two steps: first suppose $I_0 f = 0$ for $f \in C^\infty(M)$ and argue this implies $Xu = \pi_0^* f$ for some $u \in C^\infty(SM)$. Second, use the high regularity of u to show $Xu = \pi_0^* f$ is only possible if $f = 0$. The first step of obtaining a high regularity solution to the transport equation is achieved with the following *Livšic theorem.*

Theorem 3.2 (Livšic Theorem)**.** *Let $\mathcal{M}$ be a closed manifold with transitive Anosov flow generated by X and suppose $f \in C^\alpha(\mathcal{M})$ for $\alpha \in (0,1) \cup \mathbb{N} \cup \{\infty\}$ and $If = 0$. Then there exists $u \in C^\alpha(\mathcal{M})$ such that $f = Xu$.*

Livšic [Liv71] originally proved the Livšic theorem in Hölder regularity $\alpha \in (0,1)$, then Guillemin and Kazhdan [GK80a] proved smooth regularity in the case of geodesic flows on surfaces of strictly negative curvature, and [dlLMM86] proves smooth regularity in the case of general Anosov flow by using the Journé Lemma [Jou86]. The second step in achieving injectivity is showing that a smooth solution $u \in C^\infty(SM)$ to the transport equation $Xu = \pi_0 f$ forces $f = 0$. This is achieved with the following L^2 energy estimate called the *Pestov identity.*

Theorem 3.3 (Pestov identity)**.** *For a closed surface M and $u \in C^\infty(SM)$ we have the equality*

$$\|VXu\|^2_{L^2(SM)} = \|Xu\|^2_{L^2(SM)} + \|XVu\|^2_{L^2(SM)} - (KVu, Vu)_{L^2(SM)} \tag{8}$$

where K denotes the Gaussian curvature of M.

Proof of Theorem 3.3. The Pestov identity is obtained by computing the quantity $([XV,VX]u,u)_{L^2(SM)}$ in two different ways. First use repeated integration by parts to compute:

$$\begin{aligned}([XV,VX]u,u) &= ((XVVX - VXXV)u,u) \\ &= -(VVXu,Xu) + (XXVu,Vu) = \|VXu\|^2 - \|XVu\|^2.\end{aligned}$$

Now compute $([XV,VX]u,u)_{L^2(SM)}$ by using the structure equations (1) to rewrite

$$\begin{aligned}[XV,VX] &= XVVX - VXXV \\ &= VXVX + [X,V]VX - VXVX - VX[X,V] \\ &= HVX - VXH \\ &= VHX + [H,V]X - VHX - V[X,H] \\ &= -X^2 + VKV.\end{aligned}$$

Now use the above to write

$$([XV,VX]u,u) = (-X^2u,u) + (VKVu,u) = \|Xu\|^2 - (KVu,Vu).$$

The Pestov identity (8) is then obtained by setting these two expressions for $([XV,VX]u,u)_{L^2(SM)}$ equal. □

The Pestov identity was first discovered in particular cases by Mukhometov [Muh77, Muh81] and Amirov [Ami86], then Pestov and Sharafutdinov [PS88] proved a version of the above Pestov identity in coordinates for arbitrary dimension, which was reformulated into a coordinate-free form by Knieper [Kni02]. Note the Liv̌sic theorem together with the Pestov identity quickly gives injectivity of the X-ray transform for surfaces of strictly negative curvature.

Proof of Theorem 3.1 for negatively curved surfaces. Let M be a manifold of strictly negative curvature and suppose $f \in C^\infty(M)$ satisfies $I_0 f = 0$. Then by the Liv̌sic theorem, there exists $u \in C^\infty(SM)$ such that $\pi_0^* f = Xu$. Notice $VXu = V\pi_0^* f = 0$ because any pullback $\pi_0^* f$ is constant over each fiber and for negative curvature we have

$$\|XVu\|^2_{L^2(SM)} - (KVu,Vu)_{L^2(SM)} \geq 0. \tag{9}$$

Plugging this information into the Pestov identity yields $\|Xu\|_{L^2(SM)} \leq 0$. Therefore we conclude $\pi_0^* f = Xu = 0$ and so $f = 0$. □

In fact, (9) holds generally for Anosov surfaces as shown in [PSU14] using a solution to the Riccati equation constructed by E. Hopf [Hop48], so the same argument applies in the Anosov case. The key input to the above argument is the Liv̌sic theorem, which is technical to prove. Guillarmou [Gui17] recently provided a microlocal proof of the Livšic theorem, which we build towards in sections 4 and 5.

4 Meromorphic extension of Resolvents

Consider a closed manifold $\mathcal{M}$ with an Anosov flow φ_t generated by X. The properties of this Anosov flow are then encoded in the spectrum of the operator X. Consider the resolvents $R_\pm(z) \coloneqq (z \pm X)^{-1}$ of the forward and backward Anosov flows, take $f \in C^\infty(\mathcal{M})$, and consider the functions $u_\pm^{(z)} \coloneqq R_\pm(z)f$ which by definition are the solutions to the transport equation $(z \pm X)u_\pm^{(z)} = f$ with attenuation z. Observe these solutions can be written explicitly for $\operatorname{Re} z$ large by

$$R_\pm(z)f = u_\pm^{(z)} = \int_0^\infty e^{-tz}\varphi_{\mp t}^* f\,dt. \tag{10}$$

To see why the integral (10) converges for $\operatorname{Re} z$ large, note $\|\varphi_t\|_{L^2\to L^2} \le C_0 e^{Mt}$ by the semigroup property of the flow and taking $C_0 = \sup_{t\in[0,1]} \|\varphi_t^*\|_{L^2\to L^2}$, for example. Then for $\operatorname{Re} z > M$ we can compute

$$\|R_-(z)\|_{L^2\to L^2} \le \int_0^\infty e^{-t\operatorname{Re} z}\|\varphi_t^*\|_{L^2\to L^2}dt \le C_0\int_0^\infty e^{t(M-\operatorname{Re} z)}dt \le \frac{C_0}{\operatorname{Re} z - M}.$$

The same bound applies to $R_+(z)$ and we would like to meromorphically extend these resolvents using the Analytic Fredholm Theorem, which we recall below – see [Lef25] for a proof.

Theorem 4.1 (Analytic Fredholm Theorem)**.** *Take a domain $U \subset \mathbb{C}$ and suppose $A(z) : E_1 \to E_2$ is a holomorphic family of Fredholm operators over U between Banach spaces. If $A(z_0)$ is invertible for any $z_0 \in U$, we may conclude $A(z)^{-1} : E_2 \to E_1$ is a meromorphic family of bounded operators over U.*

A first try is to consider the L^2 spectrum by considering the space $\mathcal{D}_{L^2(\mathcal{M})} = \{u \in L^2(\mathcal{M}) : Xu \in L^2(\mathcal{M})\}$ designed so that we can consider the holomorphic family of bounded maps $z \pm X : \mathcal{D}_{L^2(\mathcal{M})} \to L^2(\mathcal{M})$ with codomain $L^2(\mathcal{M})$. However, this does necessarily meromorphically extend. For example, in the case the flow is volume-preserving, the L^2 spectrum is computed to be the imaginary axis $\sigma_{L^2}(X) = i\mathbb{R}$. The issue is that the operators $z \pm X : \mathcal{D}_{L^2(\mathcal{M})} \to L^2(\mathcal{M})$ are not Fredholm.

To obtain meromorphic extensions of the resolvents and consequently more spectral information, Faure and Sjöstrand [FS11] constructed anisotropic Sobolev spaces $H_\pm^s$ finely tuned to the dynamics so that H_+^s (resp. H_-^s) has microlocal regularity H^s (resp. H^{-s}) in a conic neighborhood of E_s^* and microlocal regularity H^{-s} (resp. H^s) in a conic neighborhood of E_u^*. Then define the space $\mathcal{D}_{H^s_\pm} = \{u \in H^s_\pm : Xu \in H^s_\pm\}$ and so that we can take $z \pm X : \mathcal{D}_{H^s_\pm} \to H^s_\pm$ with codomain an anisotropic Sobolev space. Importantly, $H^s_\pm$ are chosen so these maps are Fredholm over a larger domain, which yields meromorphic resolvents over a larger domain. The construction of these anisotropic Sobolev spaces and proof of the Fredholm property is technical and is beyond the scope of this document, but we record this result below.

Theorem 4.2 (Faure–Sjöstrand [FS11])**.** *There exist anisotropic Sobolev spaces $H^s_\pm$ such that the resolvents $R_\pm(z) : H^s_\pm \to H^s_\pm$ extend meromorphically over the domain $\{\operatorname{Re} z > M - Cs\}$ for some constants M and $C > 0$.*

In fact, these anisotropic Sobolev spaces may be chosen so that $H^s_+ \cap H^s_- \subset H^s(SM)$ [Lef25]. At each z_0 in the domain, this meromorphic extension promises Laurent expansion

$$R_\pm(z) = R_\pm^{\mathrm{hol}}(z) + \sum_{k=1}^{N(z_0)} \frac{(z \pm X)^{k-1}\Pi^\pm_{z_0}}{(z - z_0)^k} \tag{11}$$

where $\Pi^\pm_{z_0} := \frac{1}{2\pi i}\int_\gamma R_\pm(\zeta)d\zeta$ for small loop γ around z_0 are the spectral projections. The range of these spectral projections is called the *generalized resonant states* and is given by

$$\operatorname{Res}^{k,\infty}_\pm(z_0) := \operatorname{ran}(\Pi^\pm_{z_0}) = \{u \in H^s_\pm : (z \pm X)^{N(z_0)}u = 0\}. \tag{12}$$

The holomorphic part $R_\pm^{\mathrm{hol}}(z)$ of the resolvents has well understood wavefront set by a result of Dyatlov-Zworski [DZ16]. In particular, if $u \in C^\infty(\mathcal{M})$, then

$$\mathrm{WF}(R_+^{\mathrm{hol}}(z)u) \subset E_u^*, \qquad \mathrm{WF}(R_-^{\mathrm{hol}}(z)u) \subset E_s^*. \tag{13}$$

Next specialize to the case X generates a volume-preserving Anosov flow, which includes geodesic flows over Anosov manifolds. Then by (10) we may compute that for $\operatorname{Re} z > 0$ we have $\|R_\pm(z)\|_{L^2\to L^2} \le 1/\operatorname{Re}(z)$ and so the resolvents grow like $1/z$ as $z \to 0$, hence 0 is a simple pole; that is, $R_\pm(z) = R_\pm^{\mathrm{hol}}(z) + \Pi_0^\pm/z$. In particular, (12) then implies $\operatorname{ran}(\Pi_0^\pm)_{H^s_\pm} = \ker(X)_{H^s_\pm}$. Note that using the substitution $w = -z$ we can compute

$$\Pi_0^+ = \int_\gamma (z + X)^{-1}dz = \int_\gamma (w - X)^{-1}dw = \Pi_0^-. \tag{14}$$

Additionally note the relationship $(z + X)^* = (\overline{z} - X)$ implies $R_+^*(z) = R_-(\overline{z})$, which gives the relationship $(R_+^{\mathrm{hol}})^*(0) = R_-^{\mathrm{hol}}(0)$ where $*$ denotes the L^2 adjoint.

This meromorphic extension of the resolvent provides new proof strategies in dynamics. For example, recall a flow is *ergodic* if $\ker X|_{L^2(\mathcal{M})} = \mathbb{C}$, and a consequence of this meromorphic extension is a new proof of ergodicity for volume-preserving Anosov flows. Hopf proved ergodicity of geodesic flows over closed surfaces of strictly negative curvature, then Anosov generalized this "Hopf argument" to what is now called Anosov flows [Ano69]. Amazingly, however, this spectral approach provides a new proof of ergodicity for Anosov flows [Lef25].

5 Guillarmou's Π operator

Guillarmou [Gui17] introduced an important operator that provides a new microlocal proof of the Livšic theorem. To motivate this operator, consider an Anosov manifold M and $f \in C^\infty(SM)$ satisfying $If = 0$, and recall we are interested in finding high regularity solutions to the transport equation $Xu = f$. Two solution candidates are $u_\pm := \pm R_\pm(0)f$, which are well-defined if $\int_{SM} f d\Sigma = 0$ because in this case the pole at 0 vanishes and so we can write $u_\pm = \pm R_\pm^{\mathrm{hol}}(0)f$. These solutions $u_\pm$ only have regularity $H^s_\pm$ respectively. However, if these solutions agree by

$$0 = u_+ - u_- = (R_+^{\mathrm{hol}}(0) + R_-^{\mathrm{hol}}(0))f,$$

then the solution $u_+ = u_-$ will have high regularity $H^s_+ \cap H^s_- \subset H^s(SM)$ for any $s > 0$. This motivates studying the kernel of the operator $\Pi := R_+^{\mathrm{hol}}(0) + R_-^{\mathrm{hol}}(0)$. In general, consider a closed manifold $\mathcal{M}$ with Anosov flow φ_t preserving a smooth volume form $d\mu$ and let X be the infinitesimal generator. Taking $R_\pm(z)$ to be the meromorphic extension of the resolvents $(z \pm X)^{-1}$ as described in Section 4, we may define *Guillarmou's* Π *operator* given by the self-adjoint operator $\Pi := R_+^{\mathrm{hol}}(0) + R_-^{\mathrm{hol}}(0)$, which we characterize the kernel of below.

Theorem 5.1 (Guillarmou [Gui17])**.** *Let $\mathcal{M}$ be a closed manifold with X the generator a volume preserving Anosov flow and take $f \in H^s(\mathcal{M})$ for $s > 0$.*

(a) If $\Pi f = 0$, then $f = Xu + v$ for $u \in H^s(\mathcal{M})$ and $v = \Pi_0^+ f \in \mathbb{C}$.

(b) Conversely, if $f = Xu + v$ with $u \in H^r(\mathcal{M})$ for any $0 < r \le s$ and $v \in \mathbb{C}$, then $\Pi f = 0$.

Proof. First suppose $\Pi f = 0$ and we consider the functions $u_\pm := \pm R_\pm^{\mathrm{hol}}(0)f$ which are elements of $H^s_\pm$ respectively. To study these functions, recall that about 0 we have the Laurent expansions

$$(z \pm X)^{-1} = R_\pm(z) = R_\pm^{\mathrm{hol}}(0) + \frac{\Pi_0^\pm}{z} + O(z).$$

Multiplying both sides by $z \pm X$ and collecting like terms gives

$$\pm X R_\pm^{\mathrm{hol}}(0) = \mathrm{Id} - \Pi_0^\pm = \pm R_\pm^{\mathrm{hol}}(0) X \tag{15}$$

where the equalities on the left and right follow by multiplying $z \pm X$ on the left and right respectively. Using the left equality gives $Xu_\pm = f - \Pi_0^\pm f$, or equivalently $f = Xu_{\pm} + \Pi_0^{\pm} f$. Importantly, note the assumption implies

$$u_+ - u_- = R_+^{\mathrm{hol}}(0) + R_-^{\mathrm{hol}}(0)f = \Pi f = 0$$

and so we conclude $u_+ = u_-$; call this function u, which is then an element of $H_+^s \cap H_-^s \subset H^s(\mathcal{M})$. Finally, recall $\Pi_0^+ f = \Pi_0^- f$ by (14), so set $v = \Pi_0^{\pm} f$ which is in $H_+^s \cap H_-^s \subset H^s(\mathcal{M})$ and recall that by 0 a simple pole of $R_{\pm}(z)$, we have (12) and so $v \in \ker X$, which by ergodicity implies $v \in \mathbb{C}$, proving (a). Conversely suppose $f = Xu + v$ as in the hypotheses of (b) and first note computing the difference of the equations in (15) and using $\Pi_0^+ = \Pi_0^-$ yields

$$X\Pi = 0 = \Pi X.$$

Next, use v is constant and (10) to explicitly compute $R_{\pm}(z)v = v/z$ for $z > 0$, so $R_{\pm}^{\mathrm{hol}}(z)v \equiv 0$ for $\operatorname{Re} z > 0$ and hence for all $z \in \mathbb{C}$. Thus $\Pi v = 0$ and so $\Pi f = \Pi(Xu + v) = 0$. □

Notice Guillarmou's Π operator allows us to strengthen the Livšic theorem from arbitrarily low Sobolev regularity to Sobolev regularity as high as f. Indeed, take $f \in H^s(\mathcal{M})$ and suppose we know $f = Xu$ for $u \in H^r(\mathcal{M})$ for *any* $0 < r \leq s$. Then Theorem 5.1 implies $\Pi f = 0$, which in turn implies $f = Xu$ for $u \in H^s(\mathcal{M})$. In particular, if $f \in C^\infty(\mathcal{M})$ satisfies $If = 0$, then if we can show $f = Xu$ for $u \in H^r(\mathcal{M})$ for any $r > 0$, we may conclude $u \in C^\infty(\mathcal{M})$. For transitive Anosov flows, we can get this initial low regularity for u by taking $x_0 \in \mathcal{M}$ with dense orbit, then defining $u(\varphi_t x_0) := \int_0^t f(\varphi_s x_0)ds$ on this dense orbit, and using the Anosov closing lemma [KH95, Theorem 6.4.15] to argue this extends with Hölder regularity to $\mathcal{M}$ [Lef25]. That is, Guillarmou [Gui17] provides a microlocal proof of the Livšic theorem for volume-preserving Anosov flows while providing new Sobolev regularity results for the Livšic theorem (although some Sobolev regularity results were previous proved in [dlL01]). Note a microlocal proof of the Livšic theorem for general transitive Anosov flows was later given in [GBL23], using the Journé's lemma [Jou86] to get the initial low regularity solution to the transport equation.

Now we revisit the question of injectivity of the X-ray transform over an Anosov manifold M with flow generated by X. Let $f \in C^\infty(M)$ with $I_0 f = 0$, then by showing $\pi_0^* f = Xu$ for u of low regularity, we get $\Pi\pi_0^* f = 0$, which in turn implies $\pi_0^* f = Xu$ for $u \in C^\infty(SM)$, then the Pestov identity implies $f = 0$. Alternatively, we can interpret this proof of injectivity as factoring through the $\Pi\pi_0^*$ operator. That is, $I_0 f = 0$ implies $\Pi\pi_0^* f = 0$, and we will soon see $\Pi\pi_0^*$ is injective (over functions with average value 0), giving $f = 0$.

We next extend the domain of this continuous operator $\Pi\pi_0^* : C^\infty(M) \to D'(SM)$, which has adjoint $\pi_{0*}\Pi : C^\infty(SM) \to D'(M)$ by the self-adjointness of Π, and this adjoint can be decomposed $\pi_{0*}\Pi = \pi_{0*}R_+^{\mathrm{hol}}(0) + \pi_{0*}R_-^{\mathrm{hol}}(0)$. Note if $u \in D'(SM)$ we have

$$\mathrm{WF}(\pi_{0*}u) \subset \{(x,\xi) \in T^*M : \exists v \in S_x M, ((x,v), d\pi_0^\top \xi, 0) \in \mathrm{WF}(u)\}.$$

and by $\ker(d\pi_0) = \mathbb{V}$ we have $((x,v), d\pi_0^\top \xi, 0) \in \mathbb{X}^* \oplus \mathbb{H}^*$. However, it is an important result of Klingenberg [Kli74] and with a streamlined proof given by [Mn87] that $\mathbb{V} \cap (E_u \oplus \mathbb{X}) = \{0\}$, which implies $(\mathbb{X}^* \oplus \mathbb{H}^*) \cap E_u^* = \{0\}$ and therefore $\mathrm{WF}(R_+^{\mathrm{hol}}(0)f) = \varnothing$ by (13). The same argument applies to $R_+^{\mathrm{hol}}(0)f$ and so this adjoint is in fact continuous $\pi_{0*}\Pi : C^\infty(SM) \to C^\infty(M)$, which implies we can extend the domain of our operator $\Pi\pi_0^* : D'(M) \to D'(SM)$. In fact, a similar analysis using properties of the anisotropic Sobolev spaces gives that we may extend the domain to get a continuous map $\Pi\pi_0^* : H^{-s}(M) \to H^{-s}(SM)$ for all $s > 0$ [Gui17]. Furthermore, this operator is injective over functions with 0 spectral projection (i.e. over functions with average value 0 when the Anosov flow is mixing):

Theorem 5.2. *If M is an Anosov manifold, then $\Pi\pi_0^* : H^{-s}(M) \to H^{-s}(SM)$ has kernel $\mathbb{C}$ and is injective over functions with 0 spectral projection.*

First we introduce *Guillarmou's normal operator* $\Pi_0 \coloneqq \pi_{0*}\Pi\pi_0^*$, which Guillarmou [Gui17] showed to be an elliptic pseudodifferential operator of order -1 using key wavefront computations from Dyatlov-Zworski [DZ16] of the resolvents $R_\pm^{\mathrm{hol}}(z)$. In the case of the X-ray transform I_0 over simple manifolds, there is a related normal operator $I_0^* I_0$ that has the same principle symbol as Π_0, so Π_0 is the analog of $I_0^* I_0$ in the Anosov case.

Proof of theorem 5.2 for surfaces of negative curvature. Recall from the proof of Theorem 5.1 that $\Pi\pi_0^* v = 0$ for any $v \in \mathbb{C}$, accounting for the nontrivial kernel. Thus consider $f \in C^\infty(M)$ with $\Pi\pi_0^* f = 0$ and normalized so that $\Pi_0^+ f = 0$, so by Theorem 5.1 we have $\Pi\pi_0^* f = Xu$ for $u \in C^\infty(SM)$. But then using the Pestov identity 3.3 in the same way as in the proof of Theorem 3.1 implies $f = 0$. If we only have $f \in H^{-s}(M)$, then note $\Pi\pi_0^* f = 0$ implies $\Pi_0 f = \pi_{0*}\Pi\pi_0^* f = 0$ and so Π_0 an elliptic pseudodifferential operator implies $f \in C^\infty(M)$. $\square$

The more general theorem simply follows from a more general Pestov identity [Lef25].

6 Research Directions

6.1 Results

We consider closed surfaces M of constant negative curvature and let X generate the (Anosov) geodesic flow. Recall we can prove the injectivity of I_0 by factoring through $\Pi\pi_0^*$. That is, if $f \in C^\infty(M)$ has average value 0, then $I_0 f = 0$ implies $\Pi\pi_0^* f = 0$, which implies $f = 0$. The main result listed below is an explicit inversion formula for $\Pi\pi_0^*$ on closed surfaces of constant negative

Gaussian curvature $K < 0$. In fact, the result is slightly stronger, for we invert Guillarmou's normal operator $\Pi_0 = \pi_{0*}\Pi\pi_0^*$ on such surfaces using the operator

$$S_K f := \int_{S_x M} \int_0^\infty e^{-\sqrt{-K}\cdot t} f(\gamma_{x,v}(t)) dt dS_x(v), \quad f \in C_c^\infty(M) \tag{16}$$

where $\gamma_{x,v}(t)$ denotes the unique geodesic on M satisfying $\gamma_{x,v}(0) = x$, $\dot{\gamma}_{x,v}(0) = v$, and the measure dS_x on fiber $S_x M$ is defined as in Section 2.1. Applying this operator followed by the Laplace-Beltrami operator Δ gives the following inversion formula.

Theorem 6.1 (R.)**.** *Given a smooth function f with average value zero over a connected closed surface M of constant curvature $K < 0$ we have the inversion formula*

$$\Delta S_K \Pi_0 f = -8\pi^2 f. \tag{17}$$

Note the above theorem reduces the problem of reconstructing f from $I_0 f$ to the problem of constructing $\Pi_0 f$ from $I_0 f$ for surfaces of constant negative curvature. This X-ray transform on Anosov manifolds is analogous to the X-ray transform on manifolds with boundary defined by integrating over geodesics between the boundary and the Π_0 operator is analogous to the X-ray normal operator in this boundary case. We prove this by first inverting an "attenuated normal operator" $\Pi_0^{(z)} := \pi_{0*}\Pi^{(z)}\pi_0^*$ where the "attenuated Π operator" $\Pi^{(z)} : C^\infty(SM) \to C^\infty(SM)$ is defined by

$$\Pi^{(z)} := \int_{\mathbb{R}} e^{-|t|z} \varphi_t^* dt, \quad \operatorname{Re} z > 0. \tag{18}$$

This operator converges weakly to Guillarmou's Π operator as the attenuation coefficient z approaches 0 when acting on functions with average value zero. That is, over an Anosov manifold M with $f \in C^\infty(SM)$ we have

$$\lim_{z \to 0} \left\langle \Pi^{(z)} f, \psi \right\rangle = \langle \Pi f, \psi \rangle, \quad \psi \in C^\infty(SM). \tag{19}$$

provided $\int_{SM} f d\Sigma = 0$ for the Sasaki volume form $d\Sigma$ as defined in Section 2.1. The distributional pairing $\langle \cdot, \cdot \rangle$ above is also with respect to the Sasaki volume form. To invert this $\Pi_0^{(z)}$ operator, we extend the operator S_K of (16) to the operator

$$S_K^{(z)} f := \int_{S_x M} \int_0^\infty e^{-(z+\sqrt{-K})\cdot t} f(\gamma_{x,v}(t)) dt dS_x(v), \quad f \in C_c^\infty(M). \tag{20}$$

Furthermore, we will write $S^{(z)} = S_{-1}^{(z)}$ and $S = S_{-1}$. Before inverting $\Pi_0^{(z)}$ on closed surfaces of constant negative curvature, we first obtain the following inversion formula on the Poincaré disk, which is interesting in its own right and can be interpreted as a reconstruction formula for the "attenuated X-ray transform" operator $\Pi^{(z)}\pi_0^*$.

Theorem 6.2 (R.). *Given a compactly supported smooth function f over the Poincaré disk $\mathbb{D}$, for any $\mathrm{Re}(z) > 0$ we have the inversion formula*

$$(\Delta - z(z+1))S^{(z)}\Pi_0^{(z)} f = -8\pi^2 f. \tag{21}$$

The above theorem and corresponding proof is an extension of Helgason's inversion formula for the unattenuated X-ray transform on the Poincaré disk [Hel80, Theorem 1.14, Chapter III]. By descending to quotients of the Poincaré disk and normalizing with respect to curvature, we obtain the following inversion formula on closed manifolds of constant negative curvature. The theorem below is an intermediate step to proving Theorem 6.1, but can be interpreted as a reconstruction formula for the attenuated X-ray transform operator $\Pi^{(z)}\pi_0^*$.

Theorem 6.3 (R.). *Given a smooth function f over a connected closed surface M of constant curvature $K < 0$, then for $\mathrm{Re}(z) > 0$ we have the inversion formula*

$$(\Delta - z(z + \sqrt{-K}))S_K^{(z)}\Pi_0^{(z)} f = -8\pi^2 f. \tag{22}$$

With the above theorem we can then obtain Theorem 6.1 by taking the limit $z \to 0$, which one may expect by (19). However, verifying the continuity at 0 requires some work using microlocal techniques and transversality.

6.2 Future Work

Recall that there is an X-ray transform I_2 on symmetric 2-tensors that has applications to marked length rigidity and spectral rigidity, and so a natural question is to generalize the above inversion formulas to the case of symmetric covariant m-tensors $S^m(SM^*)$. Given a tensor $\alpha \in S^m(SM^*)$, define the pullback $\pi_m^*\alpha(v) \coloneqq \alpha(v, \cdots, v)$, then define the X-ray transform on tensors by $I_m \coloneqq I \circ \pi_m^*$. Now we can state our question.

Question 1. *How can the inversion formula of Theorem 6.1 generalize to the X-ray transform I_m on symmetric m-tensors?*

A leader in the field, Thibault Lefeuvre, asked about the the case of I_2 above with some application in mind after seeing a preprint of the paper containing the results in Section 6.1. Another natural question is to generalize these results to non-constant curvature.

Question 2. *How can the inversion formula of Theorem 6.1 generalize to non-constant curvature?*

Note that even in the case of simple manifolds, a nice inversion formula is only known in the case of constant curvature surfaces. However, this inversion formula takes the form $f + W^2 f = EI_0 f$ for a certain inversion operator E that can be found in [PSU23] and a smoothing operator W that is 0 for constant curvature. Thus this provides an approximate inversion formula in the case of non-constant curvature and furthermore, Krishman [Kri10] showed the invertibility of the operator $\mathrm{Id} + W^2$ when the metric is in a small C^3 neighborhood of a constant curvature metric, giving exact inversion formulas in such cases. Perhaps the Theorem 6.1 inversion formula can be formulated to include this same smoothing operator W, giving an approximate inversion formula for non-constant curvature.

Recall the inversion formula for $\Pi_0 f$ reduces the question of reconstructing f from $I_0 f$ to the question of reconstructing $\Pi_0 f$ from $I_0 f$. Thus a natural question is to complete this second step to achieve an inversion formula for I_0.

Question 3. *How can we reconstruct $\Pi_0 f$ from $I_0 f$ on a hyperbolic surface M?*

If possible, answering this question would likely require studying a particular hyperbolic surface M in which the structure of the closed geodesics are well understood and finding a method to find explicit closed geodesics that approximate orbits as the Anosov closing lemma promises.

Next, there is the question of extending the theory of the geodesic flow over an Anosov manifolds to more general flows such as Anosov *magnetic flows* over a surface, which represents the dynamics of a charged particle of unit mass and charge under a magnetic field, and is defined on the cotangent bundle $\pi : T^*M \to M$ of a closed surface M as follows. First, consider the Hamiltonian $H = \frac{1}{2} g_x(\xi, \xi)$ defined over T^*M (which is the same Hamiltonian as the geodesic flow). However, now consider the *twisted symplectic form* $\omega_\sigma := \omega_{\mathrm{can}} - \pi^*\sigma$ where ω_{can} is the canonical symplectic form, and σ is a 2-form representing the magnetic field. This defines a Hamiltonian system over T^*M, and the Riemannian metric allows us to identify this with TM, and these dynamics can then be restricted to the unit tangent bundle SM.

Given an Anosov magnetic flow over the unit tangent bundle $\pi_0 : SM \to M$, we may define the Guillarmou's Π operator of this Anosov flow, then we may define a normal operator $\Pi_0 := \pi_{0*} \Pi \pi_0^*$, and it is natural to examine if this normal retains the nice properties from the case of the geodesic flow.

Question 4. *Is the normal operator Π_0 a pseudodifferential operator of order -1 in the case of an Anosov magnetic flow over a Riemannian manifold? What is the principal symbol?*

The magnetic flow is prototype for volume-preserving Anosov flows and offer more complexity than simply geodesic flows, and answering the above question is the first step to further generalizing some of Guillarmou's techniques.

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