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Intuitionism and Infinity

Intuitionism is a specific philosophy of mathematics, meaning it tries to explain where mathematics comes from. In this paper I will explore the capability of this intuitionist mathematical philosophy to capture the infinite. Specifically, there are three types of infinity: the infinitely small, the infinitely large, and the continuum. I will describe the meaning of each category and address how Intuitionism captures or fails to capture each. In some ways, Intuitionism differs from classical notions of infinity. The ways in which Intuitionism succeeds to describe infinity rests on the acceptance of a unique mathematical object called a *choice sequence*, which forces the Intuitionist into accepting specific philosophical positions.

The best way to understand Intuitionism is in how it contrasts with classical philosophy. Classically, a mathematical system starts with axioms, underived *a priori* premises of what we assume to be true. Mathematicians then pair these fundamental axiomatic truths with logic to prove new mathematical statements. Intuitionism, however, rejects this classical axiomatic approach. To the intuitionist, the mental constructions of mathematicians provide the foundation of mathematics. The specifics in how the subject can perform these mental constructions are described in the “two acts of intuition”, which provides a solid basis for this philosophy. We will later use the two acts to describe the infinite.

First, we will analyze Intuitionism’s ability to describe the infinitely small. The infinitely small is simply the ability to capture any *real number*, which simply represents any possible distance. For example, two people could stand 2 meters apart, they could stand ⅔ of a meter apart, or even π (an infinitely long mathematical constant) meters apart. Intuitionism must describe each of these numbers -- 2, ⅔ and π -- in order to successfully capture the infinitely small. Intuitionism uses the first act as the first step to capturing the real numbers.

Intuitionism holds that mental constructions ground mathematics. The first act of intuition describes how the passage of time within the mind allows for such a construction. According to the first act, each moment in time consists of two pieces -- the now and the memory of the past. Figure 1 visually represents the intuitionistic moment; the dot denotes the present and the square denotes the memory of the past. Of course, a moment cannot last forever, for time has to move. As time progresses to the next moment, the first moment is not forgotten; in fact, the entirety of the first moment now rests in the past of the second moment. The past of the third moment contains the whole of the second moment (which in itself contains the first moment). This recursive process of the entirety of the previous moment moving into the memory slot of the next moment is captured visually in Figure 2. This ability to move on to the next moment while keeping all previous items in mind is analogous to counting. Each individual moment represents a separate number. We label the first moment in Figure 2 with “1”, the next moment with “2”, and the third moment with “3”. Of course, this process could continue for as long as we want, progressing “1,2,3,4,5 . . .” So, the intuition of time describes all the counting numbers.

Figure 1:

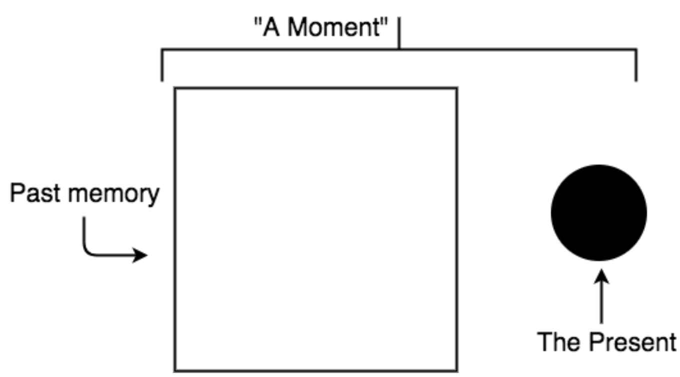
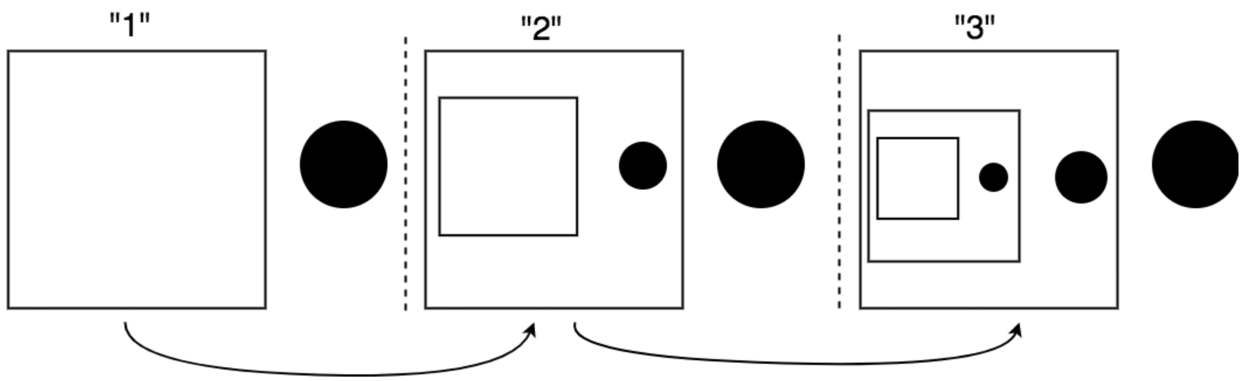


Figure 2:



While the first act successfully describes the counting numbers, it does not account for all of the infinitely small. The first act alone cannot describe distances such as ⅔ or π . The solution to capturing these missing real numbers rests in the second act of intuition. The second act describes the subject's ability to freely place previously constructed mathematical objects into sequences. For example, I can construct such a sequence out of the counting numbers that follow from the first act. Perhaps I will choose to begin my sequence with a "3", then I append on a "1", then a "4". I add a few more numbers so my sequence is "3,1,4,1,5". Currently, this sequences is only an ordinary group of numbers. In Intuitionism however, we can declare this sequence to always be unfinished, stating that we will continue to add on numbers to this sequence forever. In declaring this sequence unfinished, our sequence becomes a *choice sequence*. Choice sequences are not infinite, but they have the potential to continue for as long as we need, for we can always append on one more term. The concept of choice sequences allows Intuitionism to fully capture the infinitely small. If we declare our sample sequence to be a choice sequence, it becomes "3,1,4,1,5..." with the dots representing the unfinished property of the sequence. Notice that the initial numbers in this sequence mirror the initial digits of the constant π, which begins "3.1415..." and continues infinitely. We can translate our sequence into this decimal approximation by simply lining up all the numbers in the sequence and inserting a decimal point after the first number. If we want to, we can continue to choose numbers in our sequence to match the digits of π. In mirroring the digits of π, our sequence effectively becomes π itself. So, we have effectively captured π, a specific instance of an infinitely small number. Furthermore, there is nothing special about the number π. We can create choice sequences that represent any arbitrary real number in simply choosing the right digits. In capturing any real number, Intuitionism has succeeded in describing the infinitely small.

As an important note, the intuitionist does not consider the symbol π a number. The decimal expansion "3.1415..." is not the number in question either. To the intuitionist, the number that π represents is in the choice sequences itself, which would appear as "3,1,4,1,5...". The other versions of π are simply shorthand for the sequence.

The continuum is the expression of all the "infinitely small" numbers simultaneously. Visually, an individual infinitely small number represents a distance. Continuing with this visual, the continuum would parallel an interval of distance such as meter stick. Every point on the meter stick represents an individual distance, but the meter stick as a whole is the expression of every distance in the interval simultaneously. Because we can think of the continuum as an interval of distance, we denote it with a line spanning infinitely in both directions, which you may of heard referred to as "the number line". Just as every distance has a corresponding location on a meter stick, every real number has a place on the number line.

Intuitionists hold that an understanding of the continuum comes naturally to us through the basic intuition of time described in the first act. Recall the recursive process of the progression to future moments in time as illustrated earlier in Figure 2. While the intuition of the counting numbers is held in the individual moments, the intuition of the continuum comes from the movement from one moment to the next. The continuum is in the arrows representing the transition between the moments. However, this intuitive continuum differs from its classical counterpart. Classically, we think of a specific number as being strictly greater than, less than, or equal to a separate number which this continuum rejects. The intuitive continuum has the property of being unordered. Van Atten describes the following useful visual to grasp this difference between the two continuums. The classical continuum resembles many grains of sand lined up next to each other, and each grain of sand represents a number. If we consider a specific grain of sand, every other grain of sand must lie strictly to the right or to the left. This property parallels the understanding that every number is strictly greater than or less than a separate number. On the other hand, the intuitive continuum behaves like many strands of string cheese. Numbers can stretch and overlap with other numbers, like string cheese. This overlapping often makes it nonsensical to define one number as strictly greater than or less than the others.

This continuum as described by the intuition of time raises some concerns. Firstly, this intuitive continuum relies on an *a priori* understanding of the continuum, which appears extremely convenient. Additionally, it is easy for a challenger of Intuitionism to make the claim that they do not have this intuition, which is a difficult claim to a refute. Furthermore, this ill-defined concept is difficult to break down and analyze. Due to these numerous concerns, Brouwer developed a new theory of the continuum with choice sequences. This new theory was simply meant to more rigorously describe the intuitive continuum, so the the intuition of time still motivates the theory.

Intuitionists formally capture the intuitive continuum through a careful application of choice sequences. The subject has complete freedom over what numbers he choose to place into a choice sequence in creating it. As a consequence of this complete freedom, if the subject chooses, they can place restrictions on what numbers are allowed in the future. In other words, nothing stops the subject from consciously refusing to place specific numbers into the sequence later on. For instance, the subject can place a restriction that bans the use of even numbers. intuitionists call a set of specific restrictions on a choice sequence a *spread law*. Our subject's spread law would appear as "no even numbers can be placed into the sequence". Once a spread law is in place, some sequences are valid and some are invalid. A choice sequences beginning "1,3,5..." is a valid sequence, for it does not contain even numbers and therefore obeys our spread law. The sequence "1,2,3..." on the other hand contains the number "2", an even number, so it does not obey our spread law, making it an invalid combination. Now, imagine simultaneously creating every valid sequence. This simultaneous creation results in the *elements of the spread*. At every point in a choice sequence that we would normally make a choice, we instead entertain every possible option, creating every possibility. At every choice, our original sequence splits into multiple sequences. This splitting notion motivates an infinitely branching tree as a visual representation of the elements of the spread. Any individual path up the branches of the tree represents an individual choice sequence, but the tree as a whole is a collection of choice sequences. The continuum consists of an infinite collection of real numbers, and Intuitionism denotes each real number with a choice sequence, so a tree that holds infinitely many choice sequences should be capable of capturing the continuum. The intuitionist can in fact carefully engineer a spread law so that the resulting elements of the spread results in the continuum.

This concept of spread provides a formal description of the intuitive continuum. The natural question is: does the continuum as described by spread share the same unordered property as the Intuitive continuum? To answer this question, we require an understanding of the intuitionistic notion of truth. The origin of mathematical objects is in mental constructions. So, to the intuitionist, a mathematical statement is true only if a proof of the statement has been constructed. Similarly, a statement is false when a disproof of the statement has been constructed. A mathematical statement is neither true nor false before the construction of a proof. This undefined truth value is stronger than simply a lack of epistemic knowledge regarding the truth value. Rather, the truth value is metaphysically indeterminate. If the proof has not been constructed, the proof does not yet exist. So, in intuitionism, some statements have an indeterminate truth value. The intuitionist can then construct a choice sequence that relies on a currently indeterminate truth value. The number the choice sequence represents will be indeterminate, causing some degree of uncertainty in the value of the number. By creating numbers with uncertainty, intuitionists can engineer examples that are neither greater than, less than, or equal to other numbers. These specific examples demonstrate the compatibility of the spread continuum with the unordered nature of the original intuitive continuum.

With the continuum in mind, we are equipped to grasp the meaning of the infinitely large. In classical mathematics, there are different sizes of infinity. Firstly, we can put all of the natural numbers into an infinitely long list through the act of counting. The length of this list is infinite, specifically it is *countably infinite*, meaning we can start counting the elements of the list in a pattern that would cover the entire list given an infinite amount of time. However, the amount of elements in the continuum is *uncountably infinite* -- there is no pattern that will eventually cover all of the elements. Intuitionism accepts this distinction between countable and uncountable infinities. Classically however, there are uncountable infinities that are bigger than the amount of elements in the continuum. These bigger uncountable infinities are the infinitely large. Choice sequences cannot represent these higher infinities. Without any other tools to turn to, Intuitionism must accept its inability to capture the infinitely large. The intuitionist response to the accusation of this incapability of capturing the infinitely large is that the larger infinities "lose contact with the firm ground of mathematics" (Atten 7). In other words, the infinitely large does not exist in Intuitionism because there is no corresponding mental construction that can capture this concept, so it does not belong in mathematics.

We have looked at the ability of Intuitionism to express three different kinds of infinity: the infinitely small, the continuum, and the infinitely large. Intuitionism successfully captures the infinitely small. The intuitionist can describe the continuum, but with the different property of being unordered. However, Intuitionism has no means of explaining the infinitely large. Intuitionism does not replicate the classical notions of infinity perfectly, raising concerns of these differences causing problems. Does the unordered nature of the continuum and the inability to express the infinitely large damage the practicality of mathematics?

First of all, the intuitive continuum is unordered; does this feature of not all numbers being strictly greater than, less than, or equal to each other hurt the practicality of math? Only some numbers in the continuum are unordered. Specifically, ill-defined numbers on the continuum are unordered. The choice sequences that represent these numbers are always of a finite length, meaning that the number does not represent an individual point on the number line. Instead, intuitionistic numbers on the number line represent a small window or a strand of string cheese. Because a choice sequence makes up a window as opposed to a dot, we can always create another choice sequence with a window that *overlaps* with the window of the original sequence. It is when and only when two windows overlap that it becomes unclear whether one number is greater than, less than, or equal to the other. However, choice sequences are always unfinished. If it is unclear which number is greater than, less than, or equal to another, the creator of the sequence can easily append additional numbers onto the end. Appending additional numbers makes the sequence more accurate. This accuracy makes the window smaller and ensures the two windows no longer overlap. The "unordered" nature of the intuitive continuum means that only some numbers are unordered. The creator always has the ability to expand the choice sequences to ensure that numbers do not have this property, reserving the practicality of the continuum.

The failure of Intuitionism to capture the infinitely large has no effect on the practicality of mathematics. The concept of the infinitely large is specific to modern set theory; in fact, older and weaker versions of set theory deny the existence of the infinitely large but still function similarly. The infinitely large is primarily an acceptance of a distinction between types of uncountable infinities. The rejection of this concept represents a different philosophical view of mathematics, but preserves the practicality of mathematics.

While Intuitionism has its differences from classical mathematics, Intuitionism successfully captured the infinitely small and the continuum. However, in arriving at these infinities, the intuitionist has to accept the mathematical object of a choice sequence. These choice sequences differ from classical mathematical objects and accepting them has philosophical implications.

Choice sequences have the essential and defining property of being forever unfinished, so they rely on the progression of time. At any given point, the length of a choice sequence is finite; however, if needed, the creator of the sequence can always append on another number to increase the accuracy of the choice sequence. The choice sequence always has the *potential* to approach infinite length, allowing for the expression of the infinitely small without actually being of infinite length. However, being "unfinished" relies on time; something is only unfinished if it can be changed in the future. Choice sequences exist within time and rely on the flow of time; they are *intratemporal*. The real, physical objects we see around us are all intratemporal, but ideal objects are thought of as existing outside of time. Perhaps we develop and change our real theories of the ideal objects within time, but the ideal itself exists outside of time. However, the ideal of the choice sequences is intratemporal. So, in accepting choice sequences, the intuitionist simultaneously accepts that that at least some ideal objects exist inside of time. Accepting that ideal objects exist inside of time suggest the fundamental nature of the passage of time. If time does pass, then the intuitionist must accept a dynamic theory of time as opposed to the more popular static theories of time. So, the time dependence of choice sequences forces intuitionists to accept static theories of time.

Secondly, choice sequences are dependent on a person of some form to create them. Note that throughout this paper, there is always a person attached to the development of choice sequences; perhaps this person was myself or "we" or, perhaps it was "a creator" or "the subject". Choice sequences are unable to exist by themselves, requiring a creator to bring them into existence. Without an attached subject, there is no one with the potential to append additional numbers to the sequence. This infinite potential is essential for finite objects to capture the infinite. The inability for choice sequences, a mathematical object, to exist outside of the creator is a clear rejection of Platonism.

In Intuitionism, there is a formal concept of the *creating subject*. The creating subject is a theoretical subject with the capability to construct mathematical objects, which includes the ability to generate choice sequences. At first, it appears that this creating subject notion removes the reliance of choice sequences on a physical creator. However, most intuitionists interpret "creating subject" as replaceable with "I" or "we". The use of the creating subject was simply a tool to formally express mathematical constructions in the intuitionist version of logic. Even in rejecting the replaceability of "creating subject" with "I" or "we", choice sequences cannot be separated from a physical subject. The intuitionist would hold that a subject still needs to construct the idea of the creating subjects itself, for it is blatantly circular to claim that the creating subject creates the creating subject. So, even if it is this "creating subject" that constructs the sequences, we construct the creating subject, meaning choice sequences still depend on a physical creator.

Lastly, choice sequences rely heavily on the assumption that the subject actually has complete theoretical freedom to choose the numbers of the choice sequence. The choice sequences previously addressed have been *lawless*, the subject freely chooses numbers to place in the sequence. However, the Intuitionism also accepts *lawlike* choice sequences, sequences that follow a predetermined pattern. Lawlike sequences alone cannot arrive at every single number.

Real numbers can be infinitely long and some will inevitably be without a meaningful pattern. These patternless numbers cannot be expressed with lawlike sequences, so Intuitionism relies on lawless sequences to capture the complete continuum. However, do lawless sequences exist? Perhaps, every time the subject generates what they think is a lawless sequence, they actually follow a complicated predetermined pattern. The intuitionist responds to this accusation with a distinction between free and arbitrary, agreeing that the elements of choice sequences are not arbitrary; however, they are chosen freely. Choice sequences bring rise to the issue of whether choices can actually be made freely. In order to describe the infinite, Intuitionism must hold and defend that the choices made in these sequences are indeed free and not entirely predetermined.

In this paper, I have explored Intuitionism's ability to describe three types of infinity. We found that Intuitionism successfully captures the infinitely small, successfully describes the continuum but with the differing property of being unordered, and fails to explain the infinitely large. The ways in which intuitionism differs from the classical system has no effect on the practicality of mathematics. The success of intuitionism to capture these infinities relies on the notion of a choice sequence. These choice sequences are different from most mathematical objects in three ways: these sequences exist within time, they depend on a subject to create them, and they rely on the ability of the subject to freely choose objects. These characteristics have numerous philosophical implications that Intuitionists must accept. Specifically, the intuitionist must accept that not all mathematical objects exists outside of time, making the popular block universe theory of time unlikely. The intuitionist must believe that not all mathematical objects exist independent of the subject, which is a rejection of Platonism. And the intuitionist must hold that the subjects choices of what goes into a choice sequences are not predetermined. The simple attempt of Intuitionism to describe the infinite implies restrictive philosophical positions.

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