ORBITOPES AND APPLICATIONS TO PROTEIN FOLDING

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1. Introduction

This paper serves as an introduction of *orbitopes* to undergraduate mathematics students and aims to prepare students for understanding [1] and [3]. This paper does not contain any novel results; however, the examples given in Section 2 are well-known to be constructable, but the given constructions are my own re-discoveries. Similarly, The theorems in sections $3-6$ are well known and assumed to be true, but the presented proofs are my own re-discoveries. Section 7 summarize the novel idea in [2].

An orbitope is a geometric object resulting from a group action. Constructing an orbitope requires four ingredients.

(1) A compact group G.
(2) A real vector space V.
(3) A group action ρ acting linearly on the vector space $\rho : G \times V \to V$.
(4) An element of the vector space $x \in V$.

Applying the group action on x results in an orbit $\mathcal{O}$ in the vector space V. The resulting orbitope is the convex hull of the orbit. The above notation is uses throughout the paper.

Finite examples of orbitopes are immediately accessible (ignoring a few details) and will make up Section 2. Section 3 will touch on the representation theory and convex geometry necessary to understand the definition of an orbitope for a finite group. Section 4 will give the define orbitopes for a finite group and will include a discussioin on the existence and uniqueness of a defined orbitope. The discussion of finite group orbitopes will end with the proof that every platonic solid is an orbitope in Section 5. Section 6 will work towards applications of orbitopes by further touching on convex geometry and discussing compact groups. Finally, Section 7 will show how orbitope provide surprising information regarding the protein folding problem.

2. Examples of Finite Group Orbitopes

It is not necessary to have a rigorous understanding of the notions of convex and compact to see how examples with finite groups work out. For now, simply know that all finite groups are compact. Further note that in each of the below examples, it should be verified that the given ρ is a group action, but ignore this detail for now; claim 3.2 will provide a general proof that each given ρ is indeed a group

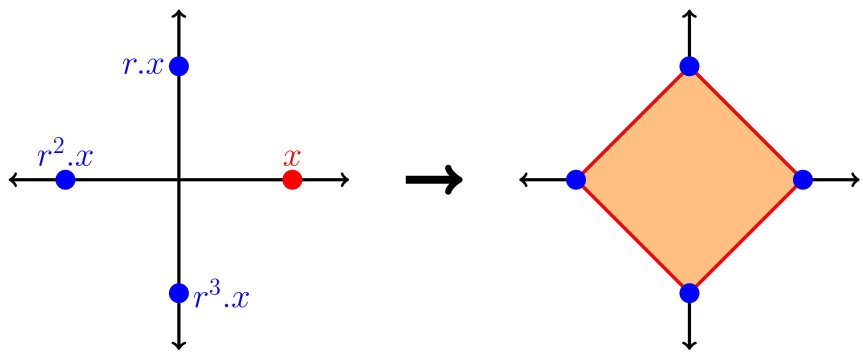


FIGURE 1. Construction of a Diamond

action. First, oberserve that a diamond can be constructed as an orbitope in the following example.

Example 2.1 (Diamond)**.** For the construction of a diamond, take the following ingredients:

(1) As the group G, take the cyclic group of order 4 with generator r. So, $G = \langle r \rangle = Z_4$
(2) Let the plane $\mathbb{R}^2$ as the real vector space V.
(3) Take the group action $\rho : Z_4 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by interpreting r as a rotation by $\pi/2$.
$$\rho : (r^n, \vec{v}) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n \vec{v}$$
(4) Finally, consider the initial element of the vector space $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Then, the action ρ on the initial value $\vec{x}$ results in an orbit within $\mathbb{R}^2$. In particular, we get:

$$\rho : (e, \vec{x}) \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \rho : (r, \vec{x}) \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \rho : (r^2, \vec{x}) \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \rho : (r^3, \vec{x}) \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

These four values account for everything in the orbit, so

$$\mathcal{O} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$$

Then, the convex hull of this set $\operatorname{conv} \mathcal{O}$ can be thought of, for the time being, as wrapping a line about the set. So, we get a diamond shape as the resulting orbitope. This process is summarized in Figure 1.

Example 2.2. As a second example, again take the vector space $V = \mathbb{R}^2$ but now let G to be D_8 – the symmetries of a square. Let $r, s \in D_8$ be the usual generators with the usual relations. Now define the action $\rho : D_8 \times \mathbb{R}^2 \to \mathbb{R}^2$ so that s represents a reflection across the x-axis and let r denote a counter clockwise rotation by $\pi/2$. As a formula,

$$\rho : (r^n s^m, \vec{v}) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^m \vec{v}$$

In this case, different orbitopes arise from different choices of the initial vector $\vec{x}$. First take $\vec{x} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$. Then, the resulting orbit is given by

$$\mathcal{O} = \left\{ \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}$$

Then the resulting orbitope is again found by taking the convex hull. Due to the eight points, the resulting orbitope is an octogon.

However, a different initial vector can result in a different orbitope. For instance, take $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to be the initial vector. Then, the resulting orbit is the same as in Example 2.1, and so a diamond is the final orbitope with this choice of initial vector.

Example 2.3 (Octahedron)**.** Now, we will consider a three dimensional example. Take $V = \mathbb{R}^3$ as the vector space, and take the symmetric group on 4 elements $G = S_4$ as the group. Note that S_4 has the following presentation.

$$S_4 = \left\langle a, b | a^2 = b^4 = (ab)^3 = 1 \right\rangle$$

This relation is satisfied by taking $b = (1234)$ and $a = (1234)(1243)^{-1}(1234)$, so the 4-cycles generate S_4. Then define the group action $\rho : Q \times \mathbb{R}^3 \to \mathbb{R}^3$ on the 4-cycles such that the element (1234) represents a $\pi/2$ rotation about the x axis while (1243) and (1324) represent $\pi/2$ rotations about the y and z axes respectively. To formally define ρ, consider the homomorphism $\varphi : S_4 \to GL(V)$ that takes these elements to these rotations.

$$\varphi : (1234) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \varphi : (1243) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \varphi : (1324) \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Repeated matrix multiplication verifies that this homomorphism preserves the relations, so this extends to a well-defined homomorphism. Then, ρ is defined by

$$\rho : (\sigma, \vec{v}) \mapsto \varphi(\sigma)\vec{v}$$

Finally, consider the initial vector $x = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. Then the following orbit follows from ρ.

$$\mathcal{O} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Intuitively, the convex hull of a three dimensional point set is given by tightly wrapping a sheet around the points with a solid interior. In this case, the resulting shape is the octahedron.

3. Background for Finite Group Orbitopes

This section gives the background necessary to understand the formal definition of orbitopes with a finite group.

3.1. **Representation Theory.**

Definition 3.1 (Representation)**.** Let G be a finite group and V a vector space over field F. Then a *representation* of G is a group homomorphism $\varphi : G \to \mathrm{GL}(V)$.

Note that in every introductory example defined above, the generators of each group G were associated with either reflection of rotation — elements of $GL(V)$. This association can be extended to a well-defined homomorphism from G to $GL(V)$ after verifying that the homomorphism satisfies the relations, giving a representation. So, in each of the above examples a representation defines a group action in the following way.

Claim 3.2. Take a group G, a vector space V, and a representation $\varphi : G \to GL(V)$. Then, the mapping $\rho : G \times V \to V$ given by $\rho : (g, \vec{v}) \mapsto \varphi(g)\vec{v}$ is a linear group action.

Proof. Denote $\rho(g, \vec{v})$ as $g.\vec{v}$. There are two things to verify: $e.\vec{v} = \vec{v}$ and $g.(h.\vec{v}) = (gh).\vec{v}$. Firstly, take $\vec{v} \in V$. Note that by definition of homomorphism, $\varphi(e) = I$ where I is the identity matrix. Then we get the desired result

$$e.\vec{v} = \varphi(e)\vec{v} = I\vec{v} = \vec{v}$$

Next take $g, h \in G$ and $\vec{v} \in V$. Note that by the definition of homomorphism we get $\varphi(g)\varphi(h) = \varphi(gh)$. The desired result follows.

$$g.(h.\vec{v}) = g.(\varphi(h)\vec{v}) = \varphi(g)\varphi(h)\vec{v} = \varphi(gh)\vec{v} = (gh).\vec{v}$$

For completeness, we will verify that this action is *linear* — a property that will be discussed formally later in Definition 4.1. Linearity in this case follows from the linearity of matrices. For $g \in G$, $\vec{v}, \vec{w} \in \mathrm{GL}(V)$, and $\alpha, \beta \in \mathbb{R}$,

$$g.(\alpha\vec{v} + \beta\vec{w}) = \varphi(g)(\alpha\vec{v} + \beta\vec{w}) = \alpha\varphi(g)\vec{v} + \beta\varphi\vec{w} = \alpha(g.\vec{v}) + \beta(g.\vec{w})$$

$\square$

The above claim and proof show that a representation of a group paired with an initial vector results in a natural orbitope construction.

3.2. **Convex Geometry.**

The definition and analysis of orbitopes requires a brief introduction to convex geometry. This section introduces basic terminology with a focus on the definition and properties of *convex hull.* Convex geometry examines subsets of a real vector space, so for the remainder of this section take V to be a real vector space.

For $x, y \in V$, denote by $[x, y]$ the line segment connecting them. Specifically, $[x, y] \coloneqq \{tx + (1 - t)y : t \in [0, 1]\}$. This allows for the definition of convex.

Definition 3.3 (Convex)**.** Given real vector space V and subset $E \subset V$, we say that E is *convex*, if for every $x, y \in E$, we have $[x, y] \subset E$.

The following brief theorem and proof is necessary in building towards convex hull.

Theorem 3.4. The intersection of any family of convex sets is itself convex.

Proof. Take two convex sets A and B and consider any two points in their intersection. By the definition of convex set, the line segment connecting the points must be in both A and B and thus lies in their intersection. $\square$

Definition 3.5 (Convex Hull)**.** Take a real vector space V and subset $E \subset V$. Then, the *convex hull* of E, denoted $\text{conv}(E)$ is the smallest convex subset containing V.

It remains to show existence and uniqueness of the convex hull. These follow from the following theorem.

Theorem 3.6. Take real vector space V, and subset $E \subset V$. Then, consider the set $\mathcal{C}$ of all convex sets containing E.

$$\mathcal{C} = \{A \subset V : A \textit{ convex, } E \subset A\}$$

Then, the convex hull of E is equivalent to the intersection of all elements in $\mathcal{C}$.

$$\text{conv}(E) = \bigcap_{A \in \mathcal{C}} A$$

Proof. First, note that the whole vector space V is convex, so $\mathcal{C}$ is nonempty which gives the existence of the intersection. Next, note that by theorem 3.4 the intersection is convex. Additionally, for every element $v \in E$, we have that $v \in A$ for all $A \in \mathcal{C}$. And so, it follows that v is in the intersection, and so E is contined within the intersection. So, the intersection is itself a convex set containing E, and by the construction it must be the minimal such set, giving us the equivalence to $\text{conv}(E)$. $\square$

Corollary 3.7. For real vector space V and subset $E \subset V$, the convex hull $\text{conv}(E)$ exists and is unique.

Proof. $\text{conv}(V)$ is equivalent to the intersection given in theorem 3.6. This intersetion exists (as noted in the above proof), and is well defined, giving uniqueness. $\square$

4. The Definition of an Orbitope

With the necessary background work behind us, I present the formal definition of an orbitope.

Definition 4.1 (Linear Group Action)**.** Take group G, real vector space V, and group action $\rho : G \times V \to V$. We call ρ *linear* if for any $g \in G$, $v, w \in V$, and $\alpha, \beta \in \mathbb{R}$ we have:

$$\rho((g, \alpha v + \beta w)) = \alpha\rho(g, v) + \beta\rho(g, w)$$

Definition 4.2 (Orbitope)**.** Take finite group G, a real vector space V, an element $x \in V$, and take a linear group action $\rho : G \times V \to V$. Then, the convex hull of the orbit of x by action ρ is an orbitope.

We now have the formal definition of an orbitope; however, it remains to show that this construction exists and defines a unique object.

Theorem 4.3. Take a compact group G, real vector space V, linear action $\rho : G \times V \to V$, and element $x \in V$. Then, the resulting orbitope as defined above exists and is unique.

Proof. G must contain identity element 1. By definition of group action, $\rho(1, x) = x$ and so x is in the resulting orbit. So, the resulting orbitope is the convex hull of a nonempty set and by Corollary 3.7, this suffices to show existence and uniqueness. $\square$

5. Platonic Solids are Orbitopes

Definition 5.1 (Platonic Solid)**.** A *Platonic solid* is a convex polyhedron in $\mathbb{R}^3$ with equivalent faces composed of congruent convex regular polygons and the same configuration at the vertices.

There are only 5 platonic solids – the tetrahedron, the cube, the octahedron, the decaheron, and the icosahedron.

Claim 5.2. Every Platonic solid is an orbitope.

Proof. Take any platonic solid S in vector space $\mathbb{R}^3$. Center the solid S at the origin and fix coordinates for every vertex of S. Let G denote the group of rotational symmetries of the platonic solid S, so

$$G = \{g \in SO(3) | g(S) = S\}$$

Note that by definition we have $G \leq \mathrm{GL}(\mathbb{R}^3)$ and so we have a homomorphism by the inclusion $i : G \to \mathrm{GL}(\mathbb{R}^3)$. Then, Theorem 3.2 applies, which gives a linear group action $\rho : G \times \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\rho(g, \vec{v}) = i(g)\vec{v} = g\vec{v}$$

Next, take the initial vector $\vec{x}$ to be given by some vertex of S. I claim that the orbit $\mathcal{O}$ of $\vec{x}$ by the action ρ is exactly the set of vertices of S.

First, take some vertex $\vec{v}$ of S. Then, consider a rotation g of the platonic solid that takes $\vec{x}$ to $\vec{v}$ and aligns the faces. By the symmetry of a platonic solid, this operation takes S to S and so $g \in G$. Then, by definition of g we have that $g.\vec{x} = g\vec{x} = \vec{v}$ and thus $\vec{v} \in \mathcal{O}$.

Next, take any point $\vec{w} \in S$ that is not a vertex of S. Then, assume for the purpose of contradiction that $\vec{w} \in \mathcal{O}$. Then, for some $g \in G$ we have $g.\vec{x} = g\vec{x} = \vec{w}$. But then g takes a vertex of S to a point that is not a vertex of S and so g cannot be a symmetry of S.

So the orbit $\mathcal{O}$ is exactly the vertices of S. I then claim that the convex hull of the orbit is exactly S. First, observe that S is convex by definition; I show that S is the smallest convex subset containing $\mathcal{O}$ by arguing that any convex set containing $\mathcal{O}$ must contain S. So, take A to be any convex set containing $\mathcal{O}$ and consider $\vec{v}, \vec{w} \in \mathcal{O}$. By definition of convex, the line segment $[\vec{v}, \vec{w}]$ must be contained in A. In other words, A contains every edge of S. Similarly, take any point $\vec{u} \in S$ contained in a face of S. Any point on a face rests inside a line segment connecting two edges and so we must have $\vec{u} \in A$. Finally, every vector contained within the orbit lies on the line connecting two faces and so must be contained in A.

Thus, every vertex, edge, face, and interior point of S is contained in A and thus $S \subseteq A$. □

Note that this proof holds when exteding the definition of Platonic solid to higher dimensions.

6. Background for Further Applications

6.1. **More Convex Geometry.** Orbitopes as presented do not appear to be of any practical use; however, the following definition and theorem demonstrate that convex hulls have a link to weighted averages and probabilities. An alternate method to define the convex hull of a set is to consider all weighted averages of vectors in the set.

Definition 6.1 (Alternate Definition of Convex Hull)**.** Take a real vector space V and a subset $E \subset V$. Then the convex hull of E is the set of all linear combination $\sum_j p_j v_j$ such that $v_j \in E$ for each j and $\sum_j p_j = 1$. A linear combination such that the scaling factors sum to one is called a *convex combination.*

Note that a convex combination is simply a weighted average. To get a feel for this definition, note that the set of convex combinations of two vectors x, y is the line segment connecting the points. Indeed, the set of convex combinations can be paramatrized by a variable t with $tx + (1-t)y$. This expression is exactly the line segment $[x, y]$ defined previously. This idea is central to the proof of equivalence of the two definition.

Proof of Equivalence of Convex Hull Definitions. Take a real vector space V and a subset $E \subset V$. Denote $\text{conv}_1(E)$ to be the convex hull of E defined by the smallest convex set containing E and let $\text{conv}_2(E)$ be the convex hull of E defined by the set of convex combinations. I claim $\text{conv}_1(E) = \text{conv}_2(E)$.

First I claim $\text{conv}_1(E) \subset \text{conv}_2(E)$. Recall that $\text{conv}_1(E)$ is the smallest convex set containg E. Further, every point in E is trivially a convex combination and so it only must be shown that $\text{conv}_2(E)$ is indeed convex. For this, let $\sum \alpha_j v_j$ and $\sum \beta_j w_j$ to be two convex combinations and thus $\sum \alpha_j$. Then parametrize the line segment connecting these two points by t.

$$t \sum \alpha_j v_j + (1-t) \sum \beta_k w_k$$

And note that ever point in the line segment is itself a convex combination by $t \sum \alpha_j + (1-t) \sum \beta_k = t + (1-t) = 1$. Thus $\text{conv}_2(E)$ is indeed convex so $\text{conv}_1(E) \subset \text{conv}_2(E)$.

For the converse, I show that a convex set containing E must contain every convex combination by the method of induction on the number of terms in the sum of a convex combination. If the convex combination has only 1 term, then the combination is an element of E and so it is trivially in every convex set containing E. Now assume that every convex combination with $n \in \mathbb{M}$ terms is in $\text{conv}_1(E)$ and consider a convex combination with $n+1$ terms.

$$\sum_{j=1}^{n+1} p_j v_j$$

Then let $d = 1$ express the convex combination slightly differently

$$\sum_{j=1}^{n+1} p_j v_j = p_{n+1}v_{n+1} + (1 - p_{n+1}) \sum_{j=1}^{n} \frac{p_j}{(1 - p_{n+1})} v_j$$

Rearranging $\sum_{j=1}^{n+1} p_j = 1$ gives $\sum_{j=1}^{n} \frac{p_j}{(1-p_{n+1})} = 1$ and by $p_{n+1} \in [0, 1]$ the rexpression of the convex combination is on the line between an element of E and a convex combination of n terms. By $\text{conv}_1(E)$ convex, the line segment must be contained within $\text{conv}_1(E)$ and thus the convex combination of $n+1$ terms is contained in the set, completing the inductive step and ultimately giving $\text{conv}_2(E) \subset \text{conv}_1(E)$. □

Theorem 6.2 (Carathéodory's theorem)**.** Take a real vector space V, a subset $E \subset V$, and consider the convex hull $\text{conv}(E)$. Then, for every vector $\vec{v}$ in the convex hull, $\vec{v}$ can be expressed by a convex combination of vectors in E where the sum has at most $dim(V) + 1$ terms.

The presented proof requires the condition $\text{conv}(E)$ bounded although the theorem is true without this requirement. However, the presented inductive proof is reminiscent of a recursive algorithm that can be used to find an explicit convex combination given by the proof.

Proof. Proceed by induction on the dimension of the vector space. For a vector space of dimension 0, every convex set is a single point and thus is expressible by the point itslef, which is a convex combination with 1 term. Now assume that the theorem holds for every real vector space of dimension $n - 1$ and consider a convex hull $\text{conv}(E)$ in a vector space V of dimension n. Next, fix a point x in some convex hull and choose a vertex $v \in E$. Take the infinite line that goes through v and x and and by the convex hull bounded, this line intersects the boundary of $\text{conv}(E)$ at a point y. Note that x then lies in the line segment connecting v and y and so $x = (1 - t)v + ty$ for some $t \in [0, 1]$. Further, the boundary of a convex hull will be a convex hull in a subspace of dimension $k \leq n - 1$ and so by assumption, y can be expressed by a convex combination $\sum_{j=1}^{n} p_j v_j$. But then, x has the following expression.

$$x = (1 - t)v + ty = (1 - t)v + t \sum_{j=1}^{n} p_j v_j$$

It follows from $\sum_{j=1}^{n} p_j = 1$ that $(1 - t) + t \sum_{j=1}^{n} p_j = 1$, which verifies that x can be expressed by the convex combination of $n + 1$ points, completing the inductive step and finishing the inductive proof. □

6.2. **Compact Groups.** The group G taken to define an orbitope is not required to be finite. A group G can be infinite so long as it is *compact*. Note the following definition of compact for a metric space.

Definition 6.3 (Compact Group)**.** A group G equipped with a metric d is *compact* if the space is closed and bounded as a metric space.

Compact infinite groups share many properties with compact finite groups, so applying the compact condition is useful. An example of a compact group follows.

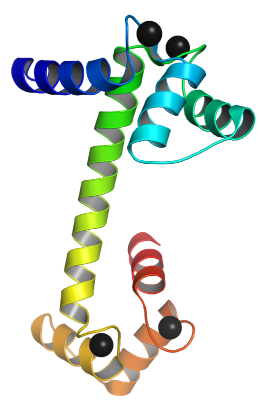
Example 6.4. Consider the special orthogonal group of order three, $SO(3)$. This is formally defined as $SO(3) = \{R \text{ is a } 3\times 3 \text{ matrix } : RR^T = \mathrm{Id}\}$. Intuitively, this is the set of rotations in three dimensional space. Next apply a norm on the set of 3×3 matrices by considering each matrix R as a 9 dimensional vector and letting the norm be defined by the standard inner product. This in turn defines a metric by $d(R_1, R_2) = \|R_1 - R_2\|$. Further, $SO(3)$ is indeed closed by noting the preimage of a closed set of a continuous function is compact and $SO(3)$ is exactly the preimage of $\{1\}$ under the (continuos) determinant function. Finally, each column of a matrix $R \in SO(3)$ must be a unit vector, and so the norm cannot get too large, giving bounded. Thus $SO(3)$ is compact.

From now on, let the word "finite" in definition 4.2 be replaced by the condition compact. However, note that the previous proof of existence of an orbitope did not require either finite compact condition, and so note that defining an orbitope does not require this compactness condition. However, having the compact condition makes the study of orbitopes far more interesting. One reason for this is that Haar's theorem gives a measure a compact group when considered as a metric space. This provides for a volume element and thus allows for integration which the following application uses.

7. Applications to Protein Folding

This section summarizes the approach and background of [2] with a focus on the ideas in the "One Metal Ion" section.

7.1. The Biology Problem. The central dogma of biology states that information goes from DNA to RNA to proteins. Proteins are made up of amino acids and the shape of a protein has an effect on the function of the protein and so biologists are interested in predicting the three dimensional shape of a protein. This is the protein folding problem.



Calmodulin

source: wikimedia commons

The paper [2] studies the shape of the protein comodulin. Comodulin has two rigid sections, the C-terminal and the N-terminal, connected by an elastic linker. Comodulin can then be modeled by two bars connected by a joint where applying a rotation $R \in SO(3)$ to the N terminal maps it onto the C terminal. [2] contributes to understanding the protein folding problem by asking the following question.

What is the probability that calmodulin is a certain orientation? That is, provide a probability distribution of the rotational states $p : SO(3) \to \mathbb{R}$.

The available measured information to answer this question follows. A paramagenetic ion can incorporated into the structure of comodulin. The paramagnetic property means that when the ion is exposed to an external magnetic field $\vec{B}$, it induces its own magnetic field $\vec{B}'$. The induced field is related to the external field by the relationship $\vec{B}' = \chi_0 \vec{B}$ where χ_0 is a 3×3 symmetric matrix called the magnetic susceptibility. Due to how this is measured, this can be further assumed to be trace-free. That is, the sum of the diagonal entries of the matrix sum to zero.

Next, fix a basis such that the C terminal lies along the x-axis. The N-teriminal then rests anywhere in space such that a rotation of R maps the N terminal onto the C terminal. When $R = \mathrm{Id}$, that is the R terminal and th C terminal overlap, let the magnetic susceptibility of the the paramagnetic ion be given by the known quantity χ_0. In general, when a rotation of R maps the N-terminal to the C-terminal, the magnetic susceptibility χ' in the coordinates is given by the change of basis formula: $\chi' = R^{-1}\chi_0 R$. This can be more simply written $\chi' = R^T \chi_0 R$ by the relation $R^{-1} = R^T$ in $SO(3)$. While each individual χ' cannot be measured, the *mean magnetic susceptibility* $\overline{\chi}$ follows from the measurement of the risidual magnetic coupling. The mean magnetic susceptibility is given by the following.

$$\overline{\chi} = \int_{SO(3)} p(R) R^T \chi_0 R d\mu \tag{1}$$

Where $p : SO(3) \to \mathbb{R}$ is a probability distribution and μ is the natural choice of area element for the group $SO(3)$.

7.2. **The Mathematical Inverse Problem.** Now, put aside the context of calmodulin and magnetic susceptibility and consider the problem as a mathematical inverse question.

Let $\overline{\chi}$ and χ_0 be known 3×3 trace-free symmetric matrices related by a probability distribution $p : SO(3) \to \mathbb{R}$ and the integral of equation 1.

Then, *what do we know about the probability distribution p?*

Solving for an entire distribution p from only two known matrix quantities is not possible. However, the following application of orbitopes give some useful insight to the quantity p. In particular, $\overline{\chi}$ can be expressed by a sum of only 6 terms.

7.3. **Applying Orbitopes to the Problem.** Construct an orbitope as follows. As the group G, take the special orthogonal group $SO(3)$. Next, let the vector space V be the space of all possible symmetric trace-free matrices. Indeed, the symmetry and trace free properties are preserved by addition and scalar multiplication, which is enough to confirm that this space is a subspace of the 3×3 matrices vector space. Further note that a symmetric 3×3 matrix has the freedom of 6 distinct numbers, but the trace-free property restricts this further to only having the free choice of 5 real numbers. Thus, this vector space is 5 dimensional. Let the group action $p : SO(3) \times V \to V$ be by conjugation:

$$p : (R, \chi) \mapsto R^T \chi R$$

Finally, let the initial vector be given by the known $\chi_0 \in V$.

As discussed previously, this defines a unique orbitope. Note that this orbitope is defined by the orbit $\left\{R^T \chi_0 R : R \in SO(3)\right\}$. Further note that the vector $\overline{\chi}$ is defined to be a weighted average by the integral in equation 1. As discussed previously, a weighted average of fintite terms is a convex combination and thus is in a convex hull. This generalizes to a weighted average over a continuum and so

$\overline{\chi}$ is in the convex hull. But then by $\dim(V) = 5$ and theorem 6.2, it follows that $\overline{\chi}$ is the sum of only 6 terms.

$$\sum_{j=1}^{6} p_j R_j^T \chi_0 R_j$$

Each R_j is a rotation and the p_j's are real numbers satisfying $\sum p_j = 1$. Note that this sum resembles integral in equation 1 and so the R_j's give insight into key rotational states. This information is motivated by an algorithm that analyzes the rotational states and gives more light to protein folding.

References

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