# Hearing the Local Orientability of Orbifolds

Sean Richardson '20, Liz Stanhope

Department of Mathematical Sciences, Lewis & Clark College

# Listening to Drums

- If you have a drum, the sound the drum makes is determined

# Listening to Drums

- If you have a drum, the sound the drum makes is determined
- However, if you hear a drum being played in another room, what can you say about it?

# Listening to Drums

- If you have a drum, the sound the drum makes is determined
- However, if you hear a drum being played in another room, what can you say about it?
- Can you hear the shape of a drum?

# Listening to Drums

- If you have a drum, the sound the drum makes is determined
- However, if you hear a drum being played in another room, what can you say about it?
- Can you hear the shape of a drum?
- Can you hear the shape of mathematical objects called "orbifolds"?

# What is an Orbifold?

# Reflectional Symmetry

- We classify symmetries with the group of isometries that leave a pattern unchanged
- Figure 1 has group $\Gamma = \{e, r\}$
- One line in space is left unmoved by all $g \in \Gamma$.
- We can fold the space into $\mathbb{R}^2/\Gamma$

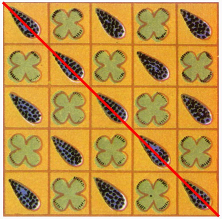

**Figure 1:** Reflectional Symmetry

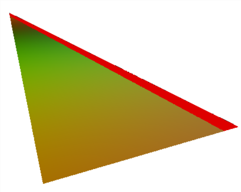

**Figure 2:** Folding the Pattern

# Rotational Symmetry

- We classify symmetries through the group of isometries that leave a pattern unchanged
- Figure 3 has group $\Gamma = \{0^\circ, 90^\circ, 180^\circ, 270^\circ\}$
- One point is left unmoved by all $g \in \Gamma$
- We can twist the space into $\mathbb{R}^n/\Gamma$ creating a party hat

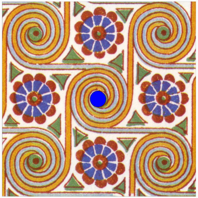

**Figure 3:** Rotational Symmetry

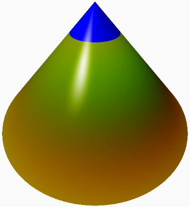

**Figure 4:** Folding the Pattern

# Orbit-folding

- We fold a pattern such that no section occurs twice.
- Result has mirror edge and cone point strata.

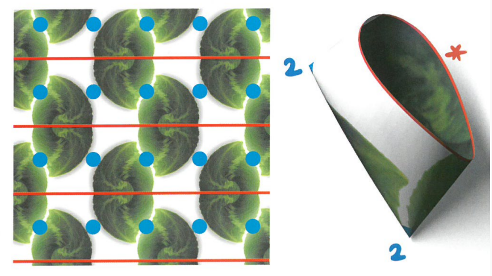


**Figure 5:** Folded Pattern from *The Symmetries of Things*

# Orbifolds

- An *orbifold* is a generalization of a manifold.
- While a manifold has local structure $\mathbb{R}^n$, an orbifold is allowed local structure $\mathbb{R}^n/\Gamma$

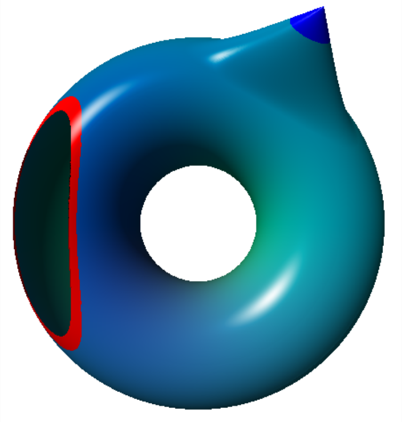

**Figure 6:** Orbifold Representation

# What is the “Sound” of an Orbifold?

# Sound Comes from Vibration

- Like physical objects, the "sound" of an orbifold is defined by how quickly it vibrates.

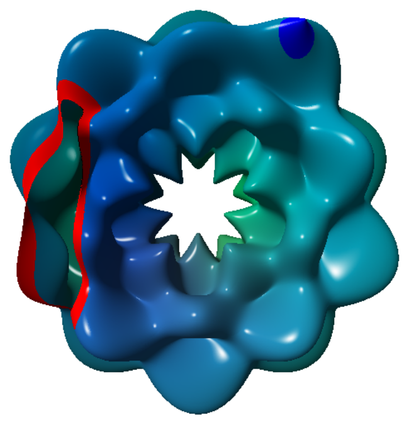

**Figure 7:** Orbifold Vibration

# The Wave Equation

- Let $u(t, \mathbf{x})$ be the amplitude of the wave at time $t$ and position $\mathbf{x}$
- The orbifold will vibrate according to the wave equation,

$$\Delta\psi = \frac{\partial^2 u}{\partial t^2}$$

- Assuming $u(t, \mathbf{x}) = A(t)\psi(\mathbf{x})$ for standing wave solutions results in

$$\Delta\psi(\mathbf{x}) = -\lambda\psi(\mathbf{x})$$

# Laplace Spectra

$$\Delta\psi(\mathbf{x}) = -\lambda\psi(\mathbf{x})$$

- Eigenvalue solutions form a discrete spectrum $\lambda_1, \lambda_2, \lambda_3, \ldots$ such that $\sqrt{\lambda_i}$ is a fundamental frequency.
- This spectrum $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called the Laplace Spectra.
- The Laplace Spectra defines "sound" on an orbifold.

# Research Question

- "Can you hear the shape of an orbifold?"
- **What properties of an unknown orbifold $\mathcal{O}$ are determined by it's known Laplace Spectra?**
    - Can you hear the types/amount of strata on an orbifold?
    - Can an orbifold and a manifold have the same Laplace Spectra?

# Heat Expansion

# Intuition of Heat Expansion

Consider:

- Heat up a point **x** on an orbifold $\mathcal{O}$
- Then, allow the heat to disperse
- The point cools down with time

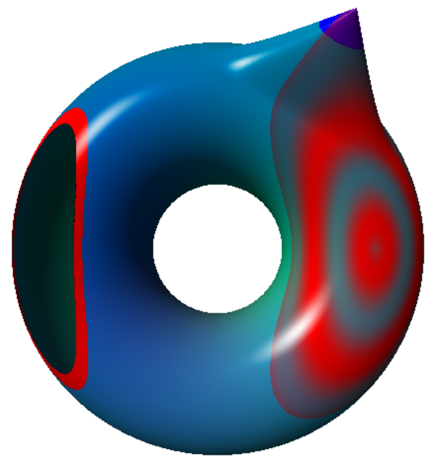

**Figure 8:** Heat Dispersing from **x**

# Intuition of Heat Expansion

We study a specific function:

- How does a point cool with time on average for every point in the orbifold $\mathcal{O}$?
- We approximate this function for small values of time.
- This function is related to the Laplace Spectrum.

# The Heat Equation

- Let $u(t, \mathbf{x})$ be the heat at time $t$ and position $\mathbf{x}$ on an orbifold $\mathcal{O}$
- Heat will spread through $\mathcal{O}$ according to the heat equation
$$\Delta u = \frac{\partial u}{\partial t}$$
- The solution to the heat equation has the form,
$$u(t, \mathbf{x}) = \int_M K(t, \mathbf{x}, \mathbf{y}) \mu_0(y) dvol M_y$$
- Consider the function $K(t, \mathbf{x}, \mathbf{y})$, the "heat kernel"

# Heat Kernel

- If heat starts at $\mathbf{p}$, there is $K(t, \mathbf{p}, \mathbf{q})$ heat at $\mathbf{q}$ at time $t$.
- Taking the trace of $K$ relates to the elements of the Laplace Spectra.

$$\mathrm{Tr}(K) = \int_M K(t, \mathbf{x}, \mathbf{x}) dvolM = \sum_{j=0}^{\infty} e^{-\lambda_j t}$$

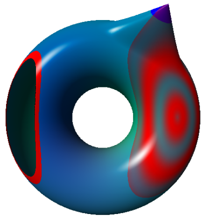

**Figure 9:** Heat Dispersing from $\mathbf{p}$

# Asymptotic Expansion of Heat Kernel

- The Heat trace has a useful approximation for small values of time.

$$\mathrm{Tr}(K) \overset{t\to 0}{\sim} \frac{a}{t} + \frac{b}{\sqrt{t}} + c + d\sqrt{t} + et + \dots$$

- Because we now know $\mathrm{Tr}(K) = \sum_j e^{-\lambda_j t}$,

$$\sum_{j=0}^{\infty} e^{-\lambda_j t} \overset{t\to 0}{\sim} \frac{a}{t} + \frac{b}{\sqrt{t}} + \dots$$

- A different value of any coefficient $a, b, c, \dots$ implies a different Laplace Spectra $\lambda_1, \lambda_2, \lambda_3, \dots$

# Asymptotic Expansion of Heat Kernel

Meaning of $\sum_{j=0}^{\infty} e^{-\lambda_j t} \overset{t\to 0}{\sim} \frac{a}{t} + \frac{b}{\sqrt{t}} + \ldots$ graphically:

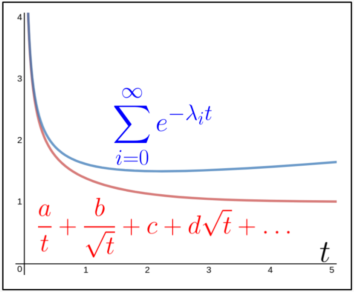


**Figure 10:** Asymptotic Expansion Approximation

# Heat Expansion Coefficients

We find the coefficients of the heat expansion through the following expression:

$$(4\pi t)^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k(\mathcal{O}) t^k$$

$$+ \sum_{N \in S(\mathcal{O})} \frac{(4\pi t)^{-\dim(N)/2}}{|\,\mathrm{Iso}(N)|} \sum_{k=0}^{\infty} t^k \int_N \sum_{\gamma \in \mathrm{Iso}^{\max}(\tilde{N})} b_k(\gamma, x) dvol_N$$

Notes:

- The strata in the orbifold affect the expansion
- Even strata and odd strata behave differently

# Result

# Local Orientability

- An isometry $g$ can preserve or reverse orientation

# Local Orientability

- An isometry $g$ can preserve or reverse orientation
- The local structure $\mathbb{R}^n/\Gamma$ of some orbifold $\mathcal{O}$ is *non-orientable* if $\Gamma$ contains a single orientation reversing isometry.

# Local Orientability

- An isometry $g$ can preserve or reverse orientation
- The local structure $\mathbb{R}^n/\Gamma$ of some orbifold $\mathcal{O}$ is *non-orientable* if $\Gamma$ contains a single orientation reversing isometry.
- We say $\mathcal{O}$ is *locally non-orientable* if $\mathcal{O}$ contains some non-orientable local structure, otherwise $\mathcal{O}$ is *orientable*.

# Local Orientability

- An isometry $g$ can preserve or reverse orientation
- The local structure $\mathbb{R}^n/\Gamma$ of some orbifold $\mathcal{O}$ is *non-orientable* if $\Gamma$ contains a single orientation reversing isometry.
- We say $\mathcal{O}$ is *locally non-orientable* if $\mathcal{O}$ contains some non-orientable local structure, otherwise $\mathcal{O}$ is *orientable*.
- Local orientability can be interpreted through the existence of a short orientation reversing path.

# We Can Hear Local Orientability

- We found that you can hear the local orientability of an orbifold $\mathcal{O}$.
- There exist no locally orientable orbifold and locally non-orientable orbifold with the same Laplace Spectra.

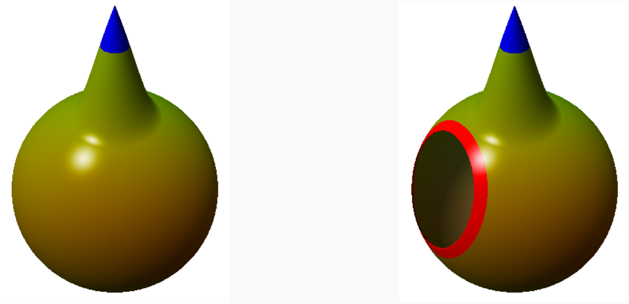

**Figure 11:** Two Orbifolds with Different Laplace Spectra

# Proof Framework

- Let $\mathcal{O}_o$ be some locally orientable odd orbifold and let $\mathcal{O}_n$ be some locally non-orientable odd orbifold.
- $\mathcal{O}_o$ will have no even strata while $\mathcal{O}_n$ will have at least one even strata.

| | $\dots$ | $t^{-1}$ | $t^{-1/2}$ | $t^0$ | $t^{1/2}$ | $t^1$ | $\dots$ |
|---|---|---|---|---|---|---|---|
| $\mathcal{O}_o$ | $\dots$ | 0 | # | 0 | # | 0 | $\dots$ |
| $\mathcal{O}_n$ | $\dots$ | $d_{-1}$ | # | $d_0$ | # | $d_1$ | $\dots$ |

**Table 1:** Heat expansion coefficients

We are able to prove that at least one $d_i$ is non-zero. So, $\mathcal{O}_o$ and $\mathcal{O}_n$ have distinct heat expansions, which implies different Laplace Spectra.

# Questions?