MATH 207A: INTRODUCTION TO DIFFERENTIAL EQUATIONS

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6/17 What is a differential equation?

The study of differential equations is responsible for almost all of the formulas in physics, chemistry, biology, engineering, economics, neuroscience, and many other fields that describe how a system evolves over time. In algebra you learned techniques to find which numbers solve various algebraic equations, which is often useful in real world situations. However, algebra only allows us to solve for a *number*, and there are many situations in which we want to find a *function* and not just a number. What will the current through a circuit be as a function of time? How can we predict the number of people infected by a disease over time? How will an object resonate at its resonance frequency? These are the types of questions that differential equations allows us to answer, and we will study these applications as well as many others throughout this course.

Example: Ponderosa pine tree. Let $y(t)$ denote the height of a Ponderosa pine tree as a function of time t. Suppose the tree grows at a rate of 1 meter per year, meaning $y'(t) = 1$. Further suppose the tree starts at a height of 2 meters, meaning $y(0) = 2$. Then what is the function $y(t)$? We can summarize these restriction on $y(t)$ as

$$\begin{cases} y'(t) = 1 \\ y(0) = 2. \end{cases} \tag{1}$$

We are therefore looking for a function $y(t)$ that has constant slope 1 and $y(0) = 2$. In this case, perhaps we can guess the solution will be the straight line $y(t) = t + 2$. We can verify this is a solution by differentiating to check $y'(t) = 1$ and plugging in $y(0) = 2$.

In this example, the "differential equation" is the function $y'(t) = 1$ and the function $y(t) = t + b$ is a "solution" to the differential equation. A ***differential equation*** is any equation that places a restriction on the derivatives of an unknown function, and a ***solution*** to a differential equation is any function that makes the differential equation true.

Example: population growth. Consider a colony of bacteria with population $B(t)$ that grows over time by cell division. Suppose the rate of growth $B'(t)$ is proportional to the population $B(t)$ (if there are twice as many cells, cell division occurs twice as frequently). For simplicity, we suppose the proportionality constant is 1 (in fact, we can always choose units so that this is true), meaning

$$B'(t) = B(t). \tag{2}$$

Let's try to guess which functions $B(t)$ satisfy this differential equations. That is, what function when differentiated is itself? One solution is e^t. However, there are actually many more solutions given by $B(t) = Ce^t$ for any real number C because

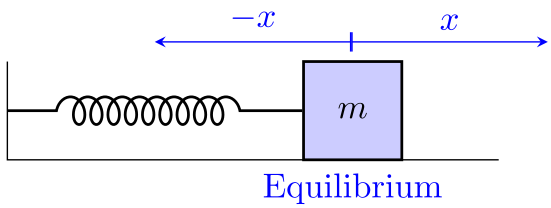
$$B'(t) = \frac{d}{dt}Ce^t = Ce^t = B(t)$$

Now suppose we also impose the "initial condition" $B(0) = 100$, meaning we begin with 100 bacteria. An ***initial condition*** specifies the value of the unknown function (or its derivatives) in the differential equation at some initial time. An ***initial value problem*** (or ***IVP*** for short) is the differential equation together with the initial condition(s). In this case, the initial value problem is

$$\begin{cases} B(t) = B(t) \\ B(0) = 100. \end{cases} \tag{3}$$

Then because $B(0) = Ce^{r\cdot 0} = C$, we require $C = 100$ to satisfy the initial condition, so the solution to the initial value problem is $B(t) = 100e^{rt}$. The hardest part in solving an initial value problem is in finding all possible solutions to a differential equation, then finding the one solution that satisfies the initial conditions is often easier. In this lecture we will try to guess the possible solutions, but much of this class will be learning techniques to derive all these possible solutions so that we do not need to rely on guessing.

Example: spring oscillation. Consider a mass on a spring as pictured below.



Let x denote the position of the position of the spring with $x = 0$ corresponding to the equilibrium position. If the mass is not in the equilibrium position, the spring exerts a force F proportional to its distance from the equilibrium position by Hooke's law, meaning

$$F = -kx \tag{4}$$

for some spring constant k. Further, Newton's law relates the force of the mass to its acceleration $x''(t)$ by

$$F = mx''. \tag{5}$$

where m is the mass of the object. For simplicity, suppose $m = 1$ and $k = 1$ so that combining (4) and (5) implies the position of the mass satisfies the differential equation

$$x''(t) = -x(t). \tag{6}$$

Let's try to guess solutions to this differential equation. That is, what function(s) when differentiated twice will be itself multiplied by -1? Both $\sin t$ and $\cos t$ have this property because

$$\frac{d^2}{dt^2}\sin t = \frac{d}{dt}\cos t = -\sin t \quad \text{and} \quad \frac{d^2}{dt^2}\cos t = \frac{d}{dt}(-\sin t) = -\cos t.$$

However, for any real number A, we can check that the function $A\cos t$ is also a solution to the differential equation (6). Similarly, $B\sin t$ is a solution to (6) for any real number B. In fact, we can add these solutions to get

$$x(t) = A\cos t + B\sin t,$$

and we can check that this $x(t)$ for any A and B is also a solution to the differential equation (6) by

$$\frac{d^2}{dt^2}(A\cos t + B\sin t) = \frac{d^2}{dt^2}A\cos t + \frac{d^2}{dt^2}B\sin t = -A\cos t - B\sin t = -(A\cos t + B\sin t).$$

Thus we have many solutions $x(t) = A\cos t + B\sin t$ because we can choose any real numbers A and B. Let's specify where the mass starts with the initial condition $x(0) = 0$. This implies

$$0 = x(0) = A\cos 0 + B\sin 0 = A.$$

So we need $A = 0$, but $B\sin t$ still satisfies this differential equation and initial condition for any B. In order to narrow this down to one solution, we need to also specify the initial velocity of the mass with an initial condition such as $x'(0) = 1$. This implies

$$1 = x'(0) = -A\cos 0 + B\cos 0 = B.$$

Setting $A = 0$ and $B = 1$ gives $x(t) = \sin t$ as the unique solution to the initial value problem

$$\begin{cases} x''(t) = -x(t) \\ x(0) = 0 \\ x'(0) = 1. \end{cases}$$

Verifying solutions to differential equations.

Problem. Verify that $y(t) = e^{-t^2}$ is a solution to the differential equation

$$y' = -2ty.$$

Solution. We just need to check

$$y'(t) = \frac{d}{dt}e^{-t^2} = -2te^{-t^2} = -2ty(t).$$

Problem. Consider the differential equation

$$t^2y'' + 3ty' + y = 0. \tag{7}$$

For what real number(s) r is the function $y(t) = t^r$ a solution to the above differential equation?

Solution. Suppose r is a real number so that $y(t) = t^r$ solves (7). This means r satisfies

$$t^2\frac{d^2}{dt^2}t^r + 3t\frac{d}{dt}t^r + y = 0.$$

By taking derivatives, this means r satisfies

$$r(r-1)t^r + 3rt^r + t^r = 0.$$

By factoring out t^r this is equivalent to

$$(r(r-1) + 3r + 1)t^r = 0.$$

The only way the above can be 0 for all values of t is if $r(r-1) + 3r + 1 = 0$ by combining like terms this is equivalent to

$$\begin{aligned} r^2 + 2r + 1 &= 0 \\ \implies \qquad (r+1)^2 &= 0 \qquad \text{by factoring.} \end{aligned}$$

Thus the only possible r so that t^r might solve (7) is $r = -1$. In order to check that t^{-1} is a solution we should compute

$$t^2\frac{d^2}{dt^2}t^{-1} + 3t\frac{d}{dt}t^{-1} + t^{-1} = 2t^{-1} - 3t^{-1} + t^{-1} = 0.$$

Types of differential equations. We say the differential equation $x''(t) = -x(y)$ is "2nd order" because there is a second derivative that appears. The equations $y'(t) = 1$ and $B'(t) = rB(t)$ we studied previously are "1st order" because the highest derivative that appears is the first derivative. In general, we say a differential equation is ***order n*** if the nth derivative is the highest derivative that appears in the equation. We will begin by studying 1st order differential equations (because 1st order is the easiest) and learning techniques to derive solutions to 1st order differential equations so that we don't have to rely on guessing solutions. In a few weeks we will move on to studying 2nd order differential equations.

A precise formula that solves a differential equation is called an ***analytic solution***. Differential equations describe an incredibly wide range of physical phenomenon, so mathematicians agree it is unrealistic to expect that every differential equation has an analytic solution. In this introductory class, we will study a few particular types of differential equations that have a precise analytic solution. In application when an analytic solution does not exist, it is often necessary to numerically approximate the solution to the differential equations and study qualitative properties of the equation, so we will also study basic numeric and qualitative analysis techniques in this class.

In particular, we will only study ***ordinary differential equations*** (or ***ODEs***) in which the unknown function only depends on a single variable. In the case the unknown function in a differential equation depends on more than one variable, we call this a ***partial differential equation*** (or ***PDE***).

6/21 Separation of variables method

Integrating to solve a differential equation. Consider the differential equation

$$\frac{dy}{dt} = 2t. \tag{8}$$

What is the solution? That is, what function $y(t)$ has derivative $2t$ everywhere? You might be able to guess that a solution is t^2 because

$$\frac{d}{dt}t^2 = 2t.$$

However, there is a systematic way to derive this solution: we can use the fundamental theorem of calculus! If $y(t)$ is a function that makes (8) true, then both sides of the equation are the same function. Therefore, if we integrate both sides from 0 to t, both sides will remain the same function. That is, we know $y(t)$ satisfies

$$\int_0^t \frac{dy}{dt}dt = \int_0^t 2t. \tag{9}$$

By the fundamental theorem of calculus, we know the left side simplifies to

$$\int_0^t \frac{dy}{dt}dt = y(t) - y(0).$$

Then integrating the right side, we find

$$\int_0^t 2tdt = t^2.$$

Therefore simplifying both sides of (9) conclude

$$y(t) - y(0) = t^2$$

Moving $y(0)$ to the right side and letting $y(0) = C$ be any constant, we find

$$y(t) = t^2 + C$$

First example of separation of variables. We can use a similar technique to solve some differential equations that include y on the right side. For example, consider the differential equation

$$\frac{dy}{dt} = t/y. \tag{10}$$

What functions $y(t)$ can solve this equation? We will use the method of "separation of variables". In this technique, we begin by rewriting the equation so that all the t's are separated out to the right and all the y's are separated out to the left:

$$y\frac{dy}{dt} = t.$$

Next we integrate both sides from 0 to t to get

$$\int_0^t y\frac{dy}{dt}dt = \int_0^t tdt.$$

We can apply a change of variables to the integral on the left to get the equivalent expression

$$\int_{y(0)}^{y(t)} ydy = \int_0^t tdt.$$

Evaluating these integrals, then making some algebraic manipulations yields

$$\begin{aligned} & \frac{1}{2}(y(t)^2 - y(0)^2) = \frac{1}{2}t^2 \\ \implies \quad & y(t)^2 - y(0)^2 = t^2 \\ \implies \quad & y(t)^2 = t^2 + y(0) \\ \implies \quad & y(t) = \pm\sqrt{t^2 + y(0)} \end{aligned}$$

By letting $C = y(0)$ be a constant, we then conclude

$$y(t) = \pm\sqrt{t^2 + C}. \tag{11}$$

Note the "±" indicates that both $y(t) = \sqrt{t^2 + C}$ and $y(t) = -\sqrt{t^2 + C}$ are candidate solutions. We can quickly verify that (11) is indeed a solution to (10) by computing

$$\frac{dy}{dt} = \frac{d}{dt} \pm \sqrt{t^2 + C} = \pm\frac{t}{\sqrt{t^2 + C}} = \frac{t}{y}.$$

Thus this technique has allowed us to find *all* the solutions to the differential equation (10)!

Shortcut technique for separation of variables. There is a popular "shortcut" for applying the method of separation of variables, which has the benefit of being faster, but comes at the cost of obfuscating the logic behind why the technique works. If we begin with the same differential equation in (10)

$$\frac{dy}{dt} = t/y.$$

Again we separate the variables, putting all the y's on the left and all the t's on the right, which again gives

$$y\frac{dy}{dt} = t.$$

We further rewrite the differential equation as

$$ydy = tdt.$$

We now take the antiderivative of both sides, so we compute

$$\int ydy = \int tdt.$$

Computing these antiderivatives and absorbing the constant into the right side yields

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + C. \tag{12}$$

We have not yet solved for $y(t)$, but we are close: the above equation related $y(t)$ to its arguments without any derivatives! Recall an ***implicit equation*** is an algebraic equation that relates the function to its arguments. We call an implicit equation that characterizes the solutions to a differential equation an ***implicit solution***. In this case (12) is an implicit solution, but we can do better by solving for y as before to get

$$y = \pm\sqrt{t^2 + C} \tag{13}$$

where we have replaced $2C$ by C (we can do this because both represent an arbitrary constant). Recall an ***explicit equation*** directly defines the function in terms of its arguments, and an explicit equation that characterizes solutions to a differential equation is called an ***explicit solution***. In this case (13) is an explicit solution. From now on we will use this faster method of separation of variables to solve differential equations, which mimics our first method while using simpler notation.

Separation of variables to solve an IVP.

Problem. Find the solution to the initial value problem

$$\begin{cases} y'(t) = -2ty \\ y(0) = 1. \end{cases} \tag{14}$$

Solution. When solving initial value problems, we start by identifying which functions can solve the differential equation. In this case, we use the method of separation of variables to the differential equation

$$\frac{dy}{dt} = -2ty. \tag{15}$$

We again manipulate this equation to move all the y's to the left and t's to the right to get

$$\frac{1}{y}\frac{dy}{dt} = -2t.$$

Following the steps in our shortcut method first yields

$$\frac{1}{y}dy = -2tdt$$

and next we get

$$\int \frac{1}{y}dy = \int -2tdt.$$

Computing these antiderivatives and remembering to add a constant gives us the implicit solution

$$\ln|y| = -t^2 + C.$$

To explicitly solve for y, exponentiate both sides by e to find

$$|y(t)| = e^{-t^2+C}.$$

We can now conclude that

$$y(t) = \pm e^{-t^2+C} = \pm e^C e^{-t^2} = Ce^{-t^2}$$

where we replaced the arbitrary constant $\pm e^C$ by the arbitrary constant C. Let's check that

$$y(t) = Ce^{-t^2} \tag{16}$$

indeed solves the differential equation by computing

$$y'(t) = \frac{d}{dt}Ce^{-t^2} = -2tCe^{-t^2} = -2ty(t).$$

Thus (16) represents exactly all possible solutions to the differential equation (15). We call an expression that represents exactly all solutions to a given differential equation a ***general solution***. Next we narrow down the general solution (16) to the solution(s) that satisfy our initial condition $y(0) = 1$ by computing

$$1 = y(0) = Ce^{-0^2} = C.$$

That is, only the solution corresponding to $C = 1$ satisfies the initial condition and so *the* solution to the initial value problem is

$$y(t) = e^{-t^2}. \tag{17}$$

A solution that is derived from the general solution after imposing a constraint (such as an initial condition) is called a ***particular solution***. In this case, (17) is the particular solution that solves our initial value problem (14).

Separable Differential Equations. Out of the following differential equations, which do you think can be solved using this "separation of variables" technique?

(a) $y'(t) = y\cos t$
(b) $y'(t) = ty + 1$
(c) $y'(t) = e^{t+y}$
(d) $y'(t) = e^{ty}$
(e) $y'(t) = y$

The primary requirement for "separation of variables" to work is that we need to rewrite the differential equation with all the y's too the left and all the t's to the right. Note we can rewrite (a) as $y^{-1}y'(t) = \cos t$, we can rewrite (c) as $e^{-y}y'(t) = e^t$ because $e^{t+y} = e^t e^y$, and we can rewrite as $y^{-1}y'(t) = 1$. However, this is not possible with (b) and (d). In general, we call a differential equation $y'(t) = f(t, y)$ ***separable*** if it can be rewritten as $g(y)y'(t) = h(t)$ for some functions $g(y)$ and $h(t)$. We found (a) is separable because we can use $g(y) = y^{-1}$ with $h(t) = \cos t$, (c) is separable because we can use $g(y) = e^{-y}$ with $h(t) = e^t$, and is separable using $g(y) = y^{-1}$ and $h(t) = 1$.

The Method of Separation of Variables. Given a general separable differential equation $y'(t) = f(t, y)$, the ***method of separation of variables*** goes as follows.

Step 1. Rewrite the differential equation as an equivalent equation of the form $g(y)y'(t) = h(t)$, which is is the same as

$$g(y)dy = h(t)dt \tag{18}$$

Step 2. Take the anti-derivative of both sides and absorb the constant into the right side. That is, we compute

$$\int g(y)dy = \int h(t)dt.$$

Then if the anti-derivative of $g(y)$ is some function $G(y)$ and the anti-derivative of $h(t)$ is some function $H(t)$ we get the equation

$$G(y) = H(t) + C \tag{19}$$

where C is an arbitrary constant. We have now solved for y implicitly.

Step 3. If you can, manipulate (19) to solve for y explicitly in terms of t and C.

Note on the logic: For an alternate explanation for why the method of separation of variables works, note that if $G(y)$ and $H(t)$ are anti-derivatives of $g(y)$ and $h(t)$ respectively, then by the chain rule and our differential equation $g(y)y'(t) = h(t)$

$$\frac{dG}{dt} = \frac{dG}{dy}\frac{dy}{dt} = g(y)y'(t) = h(t) = \frac{dH}{dt}.$$

Thus $G(y)$ and $H(t)$ have the same derivative everywhere and therefore can only differ by a constant. That is, $G(y) = H(t) + C$, demonstrating the implicit solution of (19) must be true.

This logic behind the method of separation of variables tells us that if some function $y(t)$ satisfies the differential equation $g(y)y'(t) = h(t)$, then $y(t)$ satisfies $G(y) = H(t) + C$ for some C. However the other direction is also true: suppose $y(t)$ satisfies $G(y) = H(t) + C$ for some constant C. Then because both sides are the same functions, the derivative of both sides will be the same function. That is,

$$g(y)y'(t) = \frac{d}{dt}G(y) = \frac{d}{dt}(H(t) + C) = h(t).$$

and therefore $y(t)$ satisfies the differential equation $g(y)y'(t) = h(t)$. Thus when we use the method of separation of variables, the solution we derive will satisfy the original differential equation unless we use some irreversible algebraic operation in our derivation. Thus it is generally unnecessary to check if the solution we derive satisfies the differential equation.

More Practice with Separation of Variables.

Problem. Let r be any real number. Find the general solution to

$$y' = ry. \tag{20}$$

Solution. Divide both sides by y to rewrite this differential equation as

$$\frac{1}{y}\frac{dy}{dt} = r. \tag{21}$$

That is,

$$\frac{1}{y}dy = rdt.$$

Now we compute the antiderivative of both sides, absorbing the constant into the right side. So we compute

$$\int \frac{1}{y}dy = \int dt.$$

Taking these antiderivatives and remembering the constant give us

$$\ln|y| = rt + C.$$

Now solving for y yields

$$|y| = e^{rt+C}.$$

That is, $|y| = e^C e^{rt}$, or just $|y| = Ce^{rt}$ by redefining C to be an arbitrary positive constant. Or getting rid of the absolute value and noting that $y = 0$ is a solution, we conclude the general solution to (21) is given by

$$y = Ce^{rt}$$

where C is an arbitrary constant.

Problem. Find the implicit solution to the intitial value problem

$$\begin{cases} \frac{dy}{dx} = \frac{2x}{3y^2+1} \\ y(2) = 1. \end{cases} \tag{22}$$

Solution. As usual, we begin by finding the general solution to differential equation

$$\frac{dy}{dx} = \frac{2x}{3y^2+1}.$$

We can rearrange this equation to arrive at

$$(3y^2+1)dy = 2xdx.$$

Now we compute the antiderivative of both sides, absorbing the constant into the right side. So we compute

$$\int (3y^2+1)dy = \int 2xdx,$$

which gives us the implicit general solution

$$y^3 + y = x^2 + C.$$

To find the particular solution, we use the initial condition $y(2) = 1$. That is, we plug in $x = 2$ and $y = 1$ to deduce what C must be:

$$1^3 + 1 = 2^2 + C$$

That is, $2 = C + 4$, so $C = -2$ should work.

Problem. Find the general solution to

$$\frac{dy}{dx} = e^{x-y}. \tag{23}$$

Solution. Importantly, note that this is a separable differential equation because $e^{x-y} = e^x e^{-y}$ implies we can rewrite this as

$$e^y \frac{dy}{dx} = e^x,$$

which is equivalent to

$$e^y dy = e^x dx.$$

Now we compute the antiderivative of both sides, absorbing the constant into the right side. So we compute

$$\int e^y dy = \int e^x dx,$$

which gives us the implicit solution

$$e^y = e^x + C.$$

We can solve for y to get the explicit solution

$$y(x) = \ln(e^x + C). \tag{24}$$

Let's check that the function $y(x)$ given in (24) is indeed a solution to (23). First let's compute

$$\frac{dy}{dx} = \frac{d}{dx} \ln(e^x + C) = \frac{e^x}{e^x + C}.$$

Next let's compute

$$e^{x-y} = \frac{e^x}{e^y} = \frac{e^x}{e^x + C}.$$

Thus this $y(x)$ does satisfy $\frac{dy}{dx} = e^{x-y}$ and so is a solution.

6/24 Models with exponential decay

A simple mixing problem.

Problem. Consider a container with 2 liters of salt water, which in total contains 1 gram of salt as shown in Figure 1 If the salt water in the container is pumped out at a rate of 6 liters per minute and fresh water is pumped in at a rate of 6 liters per minute, what is the mass $m(t)$ of salt (in grams) over time?

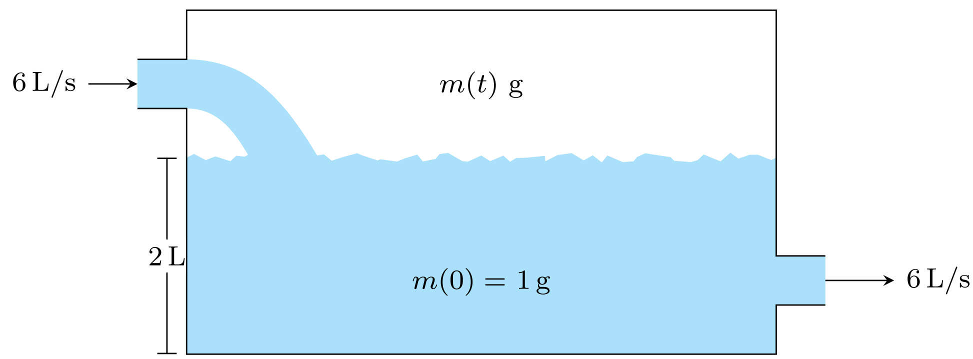


FIGURE 1. Simple mixing problem

Solution. To solve for the mass $m(t)$ of salt, we will first find the *rate of change* of $m(t)$. This allows us to write down a differential equation which we can then solve. It is always the case that

$$\text{rate of change of salt} = \text{rate of salt coming in} - \text{rate of salt going out}.$$

In this problem, there is no salt coming in, so we only need to determine the rate that salt leaves the container. If there is $m(t)$ grams of salt spread out over the 2 liter container, and 6 liters are pumped out every second, then

$$\text{salt going out in } \frac{\text{grams}}{\text{minute}} = \frac{m(t)}{2}\frac{\text{grams}}{\text{liter}} \cdot 6\frac{\text{liters}}{\text{minute}}.$$

Therefore salt leaves the container at a rate of $3m(t)$ L/min. Therefore, using liters, minutes, and grams, we have the differential equation

$$\frac{dm}{dt} = -3m \tag{25}$$

Furthermore we have the initial condition $m(0) = 1$ and we have reduced this mixing problem to solving the initial value problem

$$\begin{cases} \frac{dm}{dt} = -3m \\ m(0) = 1 \end{cases}. \tag{26}$$

The differential equation $\frac{dm}{dt} = -3m$ is a special case of (20) from yesterday, so we know that using separation of variables the general solution to (25) is

$$m(t) = Ce^{-3t}.$$

Then using the initial condition $m(0) = 1$ tells us that

$$1 = m(0) = Ce^{-3\cdot 0} = C.$$

That is, $C = 1$ and so the solution to this mixing problem is the particular solution

$$m(t) = Ce^{-3t}.$$

Some notes. Recall that the ***concentration*** is given by the total mass divided by the volume of the liquid. That is,

$$\text{concentration} = \frac{\text{mass}}{\text{volume}}$$

Recall how we calculated the rate of mass flowing out:

$$\text{mass going out} = \frac{\text{mass}}{\text{volume}} \times \text{rate water flowing out}.$$

We can conveniently rewrite this in terms of concentration as

$$\text{mass going out} = \text{concentration} \times \text{rate of water flowing out}.$$

Mixing problem with filter.

Problem. Now suppose we have a 6 liter tank of salt water initially containing 10 grams of salt as shown in Figure 2. We pump out water from the tank at a rate of 1 liter per minute, we run this water through a filter that removes half the salt, then thus same water is pumped into the tank at a rate of 1 liter per minute. What is the mass $m(t)$ of salt (in grams) over time?

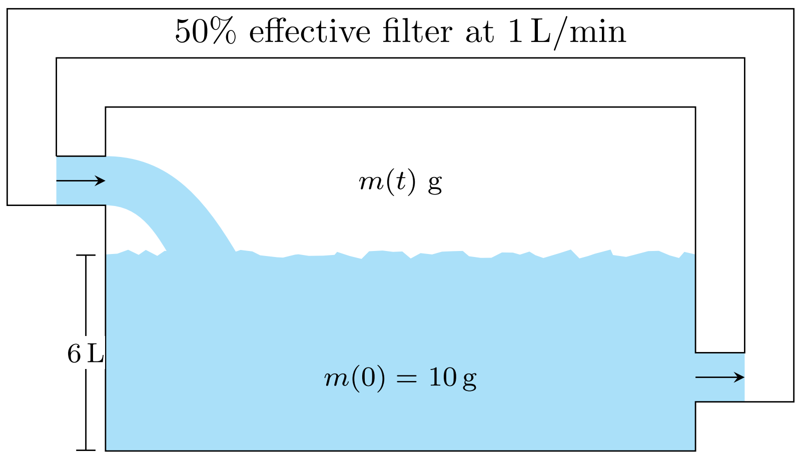


FIGURE 2. Mixing problem with filter

Solution. Again, we first determine the rate of change of the salt by

$$\text{rate of change of salt} = \text{rate of salt coming in} - \text{rate of salt going out}.$$

We can calculate the rate of salt leaving the container in the same way as in the previous example. Using units of grams, liters, and minutes we have

$$\text{rate of mass going out} = \frac{\text{mass}}{\text{volume}} \times \text{rate water flowing out} = \frac{m(t)}{6} \cdot 1.$$

Thus salt leaves at a rate of $\frac{1}{6}m(t)$. This same water going out goes through the filter, which removes half of the salt. Thus the rate of salt coming in is precisely half the rate of the salt coming out. That is,

$$\text{rate of mass coming in} = \frac{1}{2} \cdot \frac{1}{6} \cdot m(t) = \frac{1}{12}m(t).$$

Combining the salt coming in with the salt going out we arrive at the differential equation.

$$\text{rate of mass coming in} = \frac{1}{12}m(t) - \frac{1}{6}m(t) = -\frac{1}{12}m(t).$$

That is, must solve the differential equation

$$\frac{dm}{dt} = -\frac{1}{12}m(t). \tag{27}$$

We can use separation of variables to find that the general solution to this differential equation is

$$m(t) = Ce^{-\frac{1}{12}t}.$$

Using the initial condition $m(0) = 10$ we find that

$$10 = m(0) = Ce^{-\frac{1}{12}\cdot 0} = C$$

and so $C = 10$. Therefore the particular solution is

$$m(t) = 10e^{-\frac{1}{12}t}. \tag{28}$$

Changing variables. The differential equation (27) is using units of grams and minutes. How will this differential equation change if we use hours instead of minutes? When changing units in a differential equation, the first step is to relate the two quantities. In this case, if t_m is the time in minutes and t_h is the time in hours, then

$$t_m = 60t_h.$$

Our current differential equation uses minutes and says

$$\frac{dm}{dt_m} = -\frac{1}{12}m$$

Making the change of variables $t_m = 60t_h$ gives

$$\frac{dm}{d60t_h} = -\frac{1}{12}m,$$

so multiplying both sides by 60 we get the new differential equation

$$\frac{dm}{dt_h} = -5m,$$

which uses hours.

Newton's Law of Cooling. Suppose we heat up an object and place it in a room. Then the object's temperature $T(t)$ will change over time as it gradually cools. This temperature change obeys Newton's law of cooling, which states that the rate of heat loss of an object is proportional to the difference in temperatures between the object and its surroundings. That is,

$$\frac{dT}{dt} = -k(T - T_{\text{amb}})$$

where T is the temperature of the object and T_{amb} is the ambient temperature, and k is the proportionality constant which is specific to the particular object (and its surroundings): if k is large, the object cools very quickly like a baking tray and if k is small, the object cools very slowly like a teapot of boiled water.

Problem. In winter, you put some soda outside in the snow to cool down. If the soda starts at room temperature (about 68°F) and the surrounding snow is freezing (32°F), how does the temperature T of the soda evolve over time? Suppose the proportionality constant k is 2°F/hour.

Solution. By Newton's law of cooling, using units of Fahrenheit and hours we know

$$\frac{dT}{dt} = -2(T - 32). \tag{29}$$

We can solve this differential equation using the method of separation of variables. Indeed, first rearrange the differential equation (29) to get

$$\frac{1}{T-32}dT = -2dt.$$

We compute the antiderivatives of both sides; that is, compute

$$\int \frac{1}{T-32}dT = \int -2dt.$$

Computing these antiderivatives and absorbing the constant into the right side yields the implicit solution

$$\ln|T - 32| = -2t + C.$$

Now we can solve for T to get an explicit solution

$$\begin{aligned} & \ln|T-32| = -2t + C \\ \Longrightarrow \quad & |T-32| = e^{-2t+C} \\ \Longrightarrow \quad & |T-32| = e^{C}e^{-2t} \\ \Longrightarrow \quad & T-32 = Ce^{-2t} \qquad \text{by redefining C} \\ \Longrightarrow \quad & T = Ce^{-2t} + 32 \end{aligned}$$

Thus we have our general solution

$$T(t) = Ce^{-2t} + 32. \tag{30}$$

We use the initial condition $T(0) = 68$ to solve for the constant C:

$$68 = T(0) = Ce^{-2\cdot 0} + 32 = C + 32.$$

Therefore, $C = 68 - 32 = 36$. Thus we have found the particular solution

$$T(t) = 36Ce^{-2t} + 32. \tag{31}$$

Note on changing variables. In the differential equation (29) we are using units of Fahrenheit and hours. What if we wanted to use units of Celsius? Recall that if T_C is the degrees in Celsius and T_F is the degrees in Fahrenheit, the two quantities are related by

$$T_F = \frac{9}{5}T_C + 32 \qquad \text{and} \qquad T_C = \frac{5}{9}(T_F - 32).$$

Our current differential equation uses Fahrenheit and so reads

$$\frac{dT_F}{dt} = -2(T_F - 32).$$

To change to Celsius we substitute $T_F = \frac{9}{5}T_C + 32$ for all instances of T_F. This yields

$$\frac{d}{dt}\left(\frac{9}{5}T_C + 32\right) = -2\left(\left(\frac{9}{5}T_C + 32\right) - 32\right).$$

Taking the derivative on the left and simplifying the right implies

$$\frac{9}{5}\frac{dT_C}{dt} = -2\frac{9}{5}T_C$$

Then multiplying both sides by 5/9 gives us the simpler differential equation in terms of Celsius.

$$\frac{dT_C}{dt} = -2T_C \tag{32}$$

Sometimes, changing variables strategically can be useful to reducing the differential equation to something easier to manage. For example, we could even define a new variable s so that $s = 2t$. Then $t = s/2$ and so plugging this in yields

$$\frac{dT_C}{d(s/2)} = -2T_C,$$

which is equivalent to

$$\frac{dT_C}{ds} = -T_C.$$

Radioactive decay. We have seen that mixing and cooling both obey this exponential decay law. The same is true for some radioactive substance. Radioactive decay is a random process: at any given time, there is a chance that a radioactive atom decays into other atoms. Because of this, the rate of decay is proportional to the amount of radioactive molecules. Therefore if $A(t)$ is the amount of some radioactive substance, it will satisfy the following familiar differential equation

$$\frac{dA}{dt} = -kA(t)$$

where k is the proportionality constant that depends on the particular radioactive substance: k is larger if the substance decays quickly and k is smaller if the substance decays slowly.

Problem. Suppose we dump one ton of uranium inside a mountain in Nevada. After 1 billion years, it is discovered and there is still approximately 0.5 tons of non-decayed uranium. What is the approximate value of k for uranium?

Solution. First we solve the differential equation

$$\frac{dA}{dt} = -kA(t)$$

for a general k. Using the method of separation of variables, we can derive that the general solution is

$$A(t) = Ce^{-kt}.$$

Now we use the initial condition $A(0) = 1$ ton to deduce

$$1 = A(0) = Ce^{-k\cdot 0} = C.$$

That is, $C = 1$ and so the particular solution is

$$A(t) = e^{-kt}.$$

However, we still need to determine k. For this, we use our other piece of information: that $A(10^9) = 0.5$.

$$0.5 = A(10^9) = e^{-k\cdot 10^9}.$$

Now we can solve for k. Taking the natural log of both sides yields

$$-k \cdot 10^9 = \ln(0.5),$$

so

$$k = -10^{-9}\ln(0.5) \approx 7 \cdot 10^{-10}.$$

6/26 Slope fields and population dynamics

Today we will see a new way of studying solutions to differential equations without explicitly solving for the solution. Instead, we will get a visual sense of how the solutions behave.

Slope field for exponential growth. Consider a salmon hatchery with unlimited space and food so that the population $P(t)$ is roughly proportional to the growth rate of the salmon $P'(t)$. That is, the population of salmon satisfies the differential equation

$$\frac{dP}{dt} = rP(t) \tag{33}$$

for some proportionality constant r. This r measures how quickly the population multiplies: this constant would be large for rabbits, rats, and insects and would be small for elephants and turtles. Let's say $r = 1/2$ for salmon if we measure time in months, so we have the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P. \tag{34}$$

One solution to this differential equation is $P(t) = e^{t/2}$, which we can graph as in Figure **??**. One way of understanding the differential equation (34) visually is that the slope of a solution must be twice its value at that point.

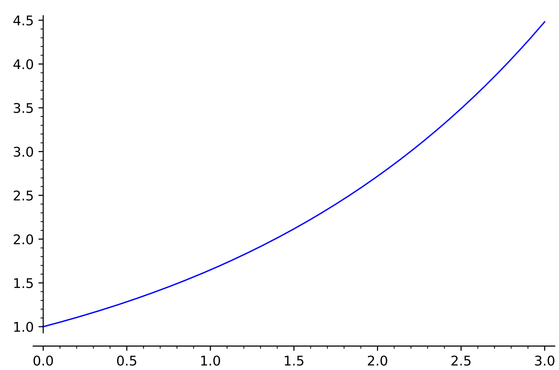


Figure 3. Solution to $P'(t) = P/2$

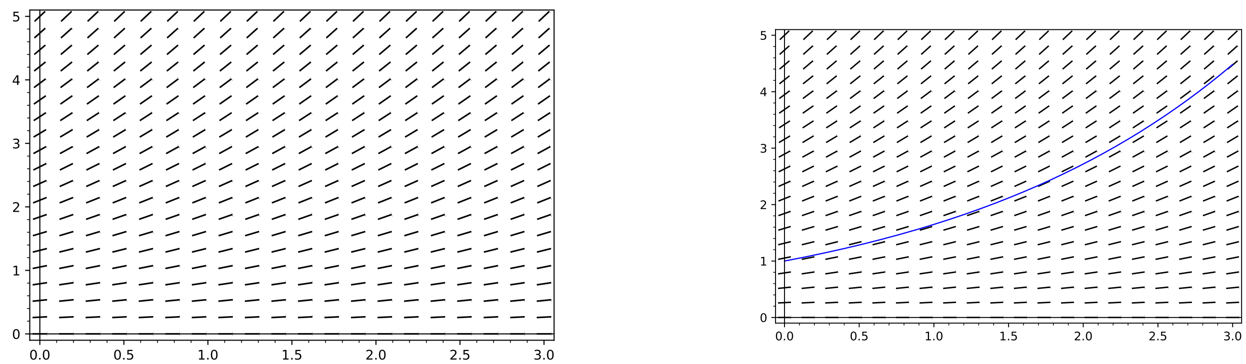
We can encode a differential equation visually with a "slope field". A ***slope field*** encodes what the slope of a solution to a differential equation must be at every point on the plane by drawing a small line segment with that same slope. For the differential equation (34), the slope field is given in Figure 4 (both with and without the solution overlayed). Solutions of a differential equation must have slopes that match the slope field at every point. In fact, even if we did not know the solution to this differential equation that satisfies $P(0) = 1$, we could estimate it by drawing a plot that roughly follows the slopes.

Slope field for exponential growth with harvesting. Consider a very small salmon farm that harvests 3 salmon every month. Then taking the same constant of proportionality $r = 1/2$, we find the salmon population satisfies the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P - 3. \tag{35}$$

Before solving for $P(t)$ let's draw the slope field, which is given in Figure 5. We can see from the slope field, that solutions look very different depending on the initial conditions. If $P(0) > 6$, we have exponential growth, but if $P(0) < 6$, the solution plummets towards 0 salmon. Importantly, if $P(0) = 6$, then the salmon population will stay at exactly 6. Indeed, we can verify that $P(t) = 6$ is a solution. Indeed,

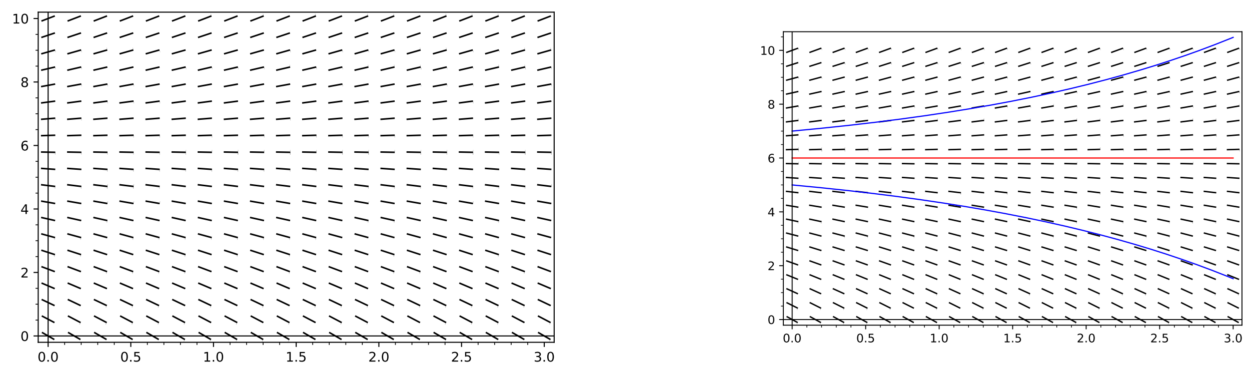
$$\frac{d}{dt}6 = \frac{1}{2}6 - 3$$



(A) Slope field

(B) Slope field with solution

FIGURE 4. Slope field for $P'(t) = P/2$



(A) Slope field

(B) Slope field with solutions

FIGURE 5. Slope field for $P'(t) = P/2 - 3$

because both sides are 0. A constant solution to a differential equation is called an ***equilibrium solution***. In this case, note that solutions that start very close to the equilibrium solution $y = 0$ get further away from the equilibrium solution. An equilibrium solution is called an ***unstable equilibrium*** if nearby solutions get farther away. We in fact could have found that $P(t) = 6$ is an equilibrium solution without drawing the slope field. An equilibrium solution $P(t)$ is constant, so $P'(t) = 0$. Therefore, an equilibrium solution to (35) must be a constant P so that

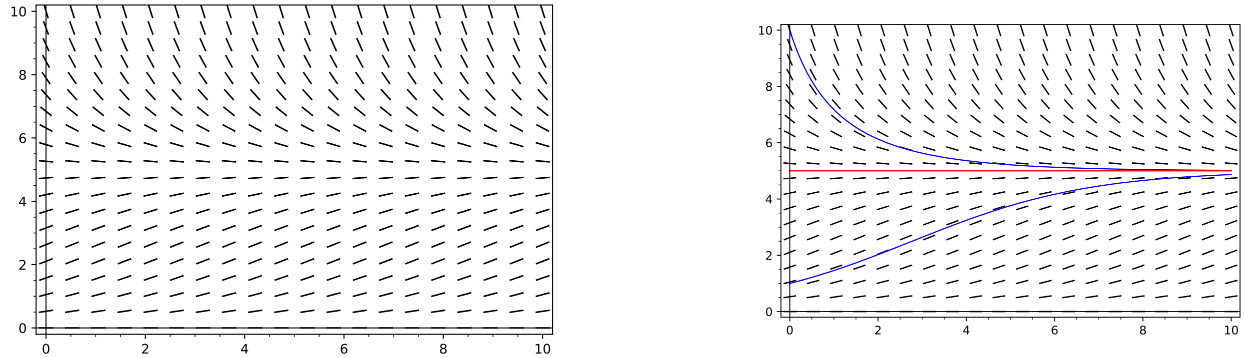
$$0 = \frac{1}{2}P - 3.$$

Then we find $P = 6$ by solving with algebra. We could also have determined that $P = 6$ is an unstable equilibrium without drawing the slope field. An unstable equilibrium $P = c$ can be precisely characterized as $P'(t) < 0$ for all P just below c and $P'(t) > 0$ for P just above c. In this case, we know $P'(t) = P/2 - 3$ and by studying the line $P/2 - 3$, we see this is true.

Slope field for logistic growth. A more accurate model of a fish population would involve a "carrying capacity". The ***carrying capacity*** is the maximum population size that the environment (space, food, water, etc.) can sustain. The ***logistic growth model*** accounts for the carrying capacity and is given by

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right). \tag{36}$$

where M is the carrying capacity of the fish and r is the growth constant. Note that if the fish population is a very small fraction of the carrying capacity, then the $1 - P/M$ term is nearly 1 and so the rate of increase of the fish is essentially the normal rate rP. However, if P is very close to the carrying capacity,



(A) Slope field

(B) Slope field with solutions

FIGURE 6. Slope field for (37)

then the $1 - P/M$ term is very close to 0 and so the rate of increase of the fish drops off dramatically. For example, say the population is 90% of the carrying capacity, so $P = 0.9M$. Plugging this in to (36) we find $\frac{dP}{dt} = 0.1rP$. That is, the fish multiply at only 10% of their usual rate because only 10% of the usual resources are available.

Suppose $M = 5$, and $r = 1/2$ so that we are studying the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P\left(1 - \frac{P}{5}\right). \tag{37}$$

The slope field to the differential equation (37) is drawn in Figure 6. Note that in this case, the constant solution $P(t) = 5$ is an equilibrium solution. Note that this equilibrium solution seems to "attract" nearby solutions. An equilibrium solution is called a ***stable equilibrium*** if nearby solutions get increasingly closer. Once again, we could have determined this is an equilibrium solution without making a slope field by solving for when $P'(t) = 0$. The right side of (37) is 0 precisely when $P = 0$ and $P = 5$, so in fact $P = 0$ is another equilibrium solution. We can also determine when an equilibrium is stable without drawing a slope field: $P = c$ is a stable equilibrium precisely when $P' > 0$ for P just below c and $P' < 0$ for P just above c. A systematic way of determining the sign of $P' = P/2(1 - P/5)$ for any value of P is to determine the signs of $P/2$ and $1 - P/5$ and then multiply the results as in the following table.

	$P < 0$	$P = 0$	$0 < P < 5$	$P = 5$	$P > 5$
$P/2$	< 0	$= 0$	> 0	> 0	> 0
$1 - P/5$	> 0	> 0	> 0	$= 0$	< 0
$P/2(1 - P/5)$	< 0	$= 0$	> 0	$= 0$	< 0

By $P' = P/2(1 - P/5)$, we see that $P' < 0$ for $P < 0$ and $P' > 0$ for $0 < P < 5$, we can determine that 0 is an unstable equilibrium. Similarly, by $P' > 0$ for $0 < P < 5$ and $P' < 0$ for $P > 5$ we can conclude 5 is a stable equilibrium.

We can in fact solve the logistic growth differential equation (36) explicitly by separating variables. First, however, let's change variables so that we are working with a simpler differential equation. First define $Q = P/M$, so that substituting $P = QM$ everywhere yields

$$\frac{d}{dt}(QM) = rQM(1 - Q),$$

which is equivalent to

$$\frac{dQ}{dt} = rQ(1 - Q) \tag{38}$$

Now let's solve (38) using separation of variables. First rearrange to get

$$\frac{1}{Q(1-Q)}dQ = rdt.$$

Then take the anti-derivative of both sides to get

$$\int \frac{1}{Q(1-Q)}dQ = \int rdt. \tag{39}$$

To compute the left side, we need to use the method of partial fractions. That is, we need to write

$$\frac{1}{Q(1-Q)} = \frac{A}{Q} + \frac{B}{1-Q}$$

for some choices of functions A and B. By cross-multiplying the right sides, we find that we need

$$\frac{1}{Q(1-Q)} = \frac{A(1-Q)+BQ}{Q(1-Q)}$$

and so we need $1 = A(1-Q) + BQ$. The choice of A and B that makes this work is $A = 1$ and $B = 1$. Therefore we can solve our integral by

$$\int \frac{1}{Q(1-Q)}dQ = \int \frac{1}{Q} + \frac{1}{1-Q}dQ = \int \frac{1}{Q}dQ + \frac{1}{1-Q}dQ = \ln|Q| - \ln|1-Q| + C = \ln\left|\frac{Q}{1-Q}\right| + C$$

Thus we found the anti-derivative on the left side of (39). Taking the anti-derivative of the right side and absorbing the constant into the right gives us the implicit solution

$$\ln\left|\frac{Q}{1-Q}\right| = rt + C.$$

Raise both sides to the power of e to get

$$\left|\frac{Q}{1-Q}\right| = e^{rt+C} = e^C e^{rt}.$$

Redefining C as $\pm e^C$ gives us

$$\begin{aligned} & \frac{Q}{1-Q} = Ce^{rt} \\ \implies\quad & Q = C(e^{rt} - Qe^{rt}) \\ \implies\quad & Q(1+Ce^{rt}) = Ce^{rt} \\ \implies\quad & Q = \frac{Ce^{rt}}{1+Ce^{rt}} \end{aligned}$$

Therefore we have an explicit general solution for Q. At this point, we can substitute $Q = P/M$ back in to solve for P:

$$\frac{P}{M} = \frac{Ce^{rt}}{1+Ce^{rt}}.$$

Thus we get the general solution

$$P = \frac{MCe^{rt}}{1+Ce^{rt}}. \tag{40}$$

Slope field for logistic growth with harvesting. Suppose that we use a logistic growth model for the population of salmon where M is the carrying capacity and r is the growth constant. However, we also account for the hatchery to harvest the fish at a constant rate H. Then the fish population satisfies the differential equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) - H. \tag{41}$$

Let's suppose $M = 8$, $r = 2$, and $H = 3$ so that we study the differential equation

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{8}\right) - 3. \tag{42}$$

Let's try to predict the equilibrium solutions before looking at the slope field. To find the constant solutions, we need to solve for P such that

$$0 = 2P\left(1 - \frac{P}{8}\right) - 3.$$

Luckily, we can factor the right side by

$$2P\left(1 - \frac{P}{8}\right) - 3 = -\frac{1}{4}P^2 + 2P - 3 = -\frac{1}{4}(P^2 - 8P + 12) = -\frac{1}{4}(P-2)(P-6)$$

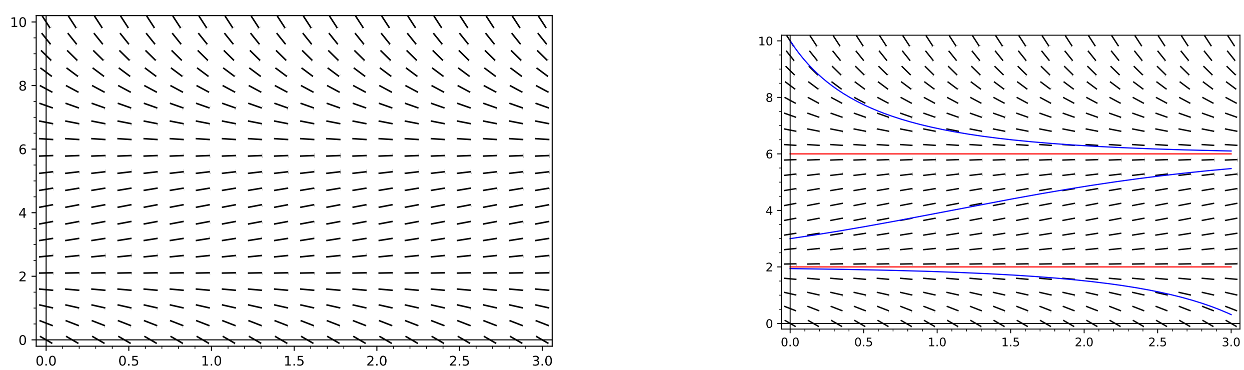
and so (42) factors to

$$\frac{dP}{dt} = -\frac{1}{4}(P-2)(P-6). \tag{43}$$

Therefore the equilibrium solutions are the constants such that $0 = -\frac{1}{4}(P-2)(P-6)$, which will be $P = 2$ and $P = 6$. Next we determine the sign of $\frac{dP}{dt}$ for different values of P by the following table.

	$P < 2$	$P = 2$	$2 < P < 6$	$P = 6$	$P > 6$
$-\frac{1}{4}(P-2)$	> 0	$= 0$	< 0	< 0	< 0
$(P-6)$	< 0	< 0	< 0	$= 0$	> 0
$-\frac{1}{4}(P-2)(P-6)$	< 0	$= 0$	> 0	$= 0$	< 0

Because the derivative is negative just below $P = 2$ and positive just above $P = 2$, we have determined that $P = 2$ is an unstable equilibrium. Similarly, because the derivative is positive just below $P = 6$ and negative just above $P = 6$, we have determined that $P = 6$ is a stable equilibrium. Note this information allows us to roughly draw what the slope field should look like, which is corroborated by the actual slope field shown in Figure 7.



(A) Slope field

(B) Slope field with solutions

Figure 7. Slope field for (42)

A note on the analytic solutions. We can use the substitution of variables $Q = P - 2$ and $P = Q + 2$ to reduce (43) to the differential equation

$$\frac{dQ}{dt} = -\frac{1}{4}Q(Q-4).$$

Or equivalently,

$$\frac{dQ}{dt} = Q\left(1 - \frac{Q}{4}\right).$$

But this is just the differential equation for logistic growth where $M = 4$ and $r = 1$. Thus by (40) thus has the general solution

$$Q(t) = \frac{4Ce^{rt}}{1 + Ce^{rt}}.$$

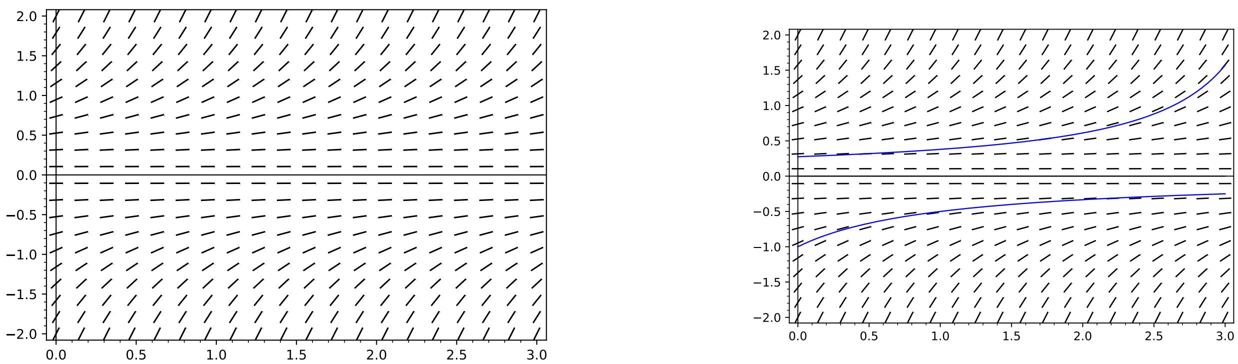
Substituting back $Q = P - 2$ we conclude that the general solution for P is

$$P(t) = \frac{4Ce^{rt}}{1 + Ce^{rt}} + 2.$$

Semistable equilibrium. Note that not all equilibrium solutions are stable or unstable. For example, consider the differential equation

$$y'(t) = y^2. \tag{44}$$

To solve for equilibrium solutions, we must solve $0 = y^2$, so we see that $y = 0$ is the only equilibrium solution. However, note that $y' = y^2 > 0$ for y just less than 0 *and* $y' = y^2 > 0$ for y just greater than 0. Thus this equilibrium is neither stable nor unstable. We call an equilibrium solution ***semistable*** if it is neither stable not unstable. The slope field for (44) is given in Figure 8.



(A) Slope field

(B) Slope field with solutions

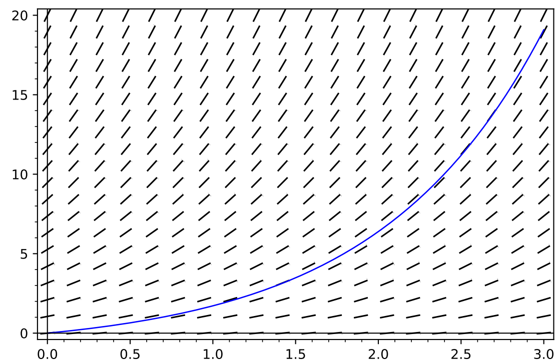
Figure 8. Slope field for (44)

6/28 Euler's method

Approximating with slope fields. Note that we can approximate solutions to differential equations by following the direction fields. For example, consider the initial value problem

$$\begin{cases} y' = y + 1 \\ y(0) = 0. \end{cases}$$

Then we can draw the slope field and roughly approximate the solution as follows We can numerically



approximate this process by using the ***tangent line approximation***

$$y(t + \Delta t) \approx y(t) + y'(t)\Delta t. \tag{45}$$

Then because $y(0) = 0$ and $y'(0) = y(0) + 1 = 1$ we can approximate

$$y(1) \approx y(0) + y'(0) \cdot 1 = 0 + 1 = 1.$$

Now taking $y(1) = 1$ and $y'(1) = y(1) + 1 = 2$ we can approximate

$$y(2) \approx y(1) + y'(1) \cdot 1 = 1 + 2 \cdot 1 = 3.$$

Applying one more step, we take $y(2) = 3$ and $y'(2) = y(2) + 1 = 4$ to approximate

$$y(3) \approx y(2) + y'(2) \cdot 1 = 3 + 4 = 7.$$

It is useful to keep track of these findings in a table

t	0	1	2	3
y	0	1	3	7

This approximation technique is called "Euler's method", which I'll now outline more precisely.

Euler's Method. ***Euler's method*** is a numerical method for approximating solutions to an initial value problem that goes as follows. Given some IVP

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases} \tag{46}$$

where t_0 is the initial time, y_0 is the initial value, and $f(t, y)$ is some function of t and y. Then we fix a ***step size*** Δt and apply the tangent line approximation

$$y(t_0 + \Delta t) \approx y(t_0) + y'(t_0)\Delta t = y(t_0) + f(t_0, y(t_0))\Delta t$$

to approximate $y(t_0 + \Delta t)$. Then we repeat, using this approximate value of $y(t_0 + \Delta t)$ to approximate

$$y(t_0 + 2\Delta t) \approx y(t_0 + \Delta t) + y'(t_0 + \Delta t)\Delta t = y(t_0 + \Delta t) + f(t_0 + \Delta t, y(t_0 + \Delta t))\Delta t.$$

Again we repeat, using this approximate value of $y(t_0 + 2\Delta t)$ to approximate

$$y(t_0 + 3\Delta t) \approx y(t_0 + 2\Delta t) + y'(t_0 + 2\Delta t)\Delta t = y(t_0 + 2\Delta t) + f(t_0 + 2\Delta t, y(t_0 + 2\Delta t))\Delta t.$$

Then we can repeat this process to approximate $y(t)$ for t as high as we would like.

Problem. Apply Euler's method with step size 0.5 to the following IVP over $[0, 2]$.

$$\begin{cases} y' = 4y^2 \\ y(0) = 1 \end{cases}$$

Solution. First we approximate

$$y(0.5) \approx y(0) + y'(0) \cdot 0.5 = y(0) + 2y(0)^2 \cdot 0.5 = 1 + 2 \cdot 1^2 \cdot 0.5 = 2.$$

Then we repeat, using this approximation $y(0.5) = 2$ to approximate

$$y(1) \approx y(0.5) + y'(0.5) \cdot 0.5 = y(0.5) + 2y(0.5)^2 \cdot 0.5 = 2 + 2 \cdot 2^2 \cdot 0.5 = 6.$$

Repeat, using use this approximation $y(1) = 6$ to approximate

$$y(1.5) \approx y(1) + y'(1) \cdot 0.5 = y(1) + 2y(1)^2 \cdot 0.5 = 6 + 2 \cdot 6^2 \cdot 0.5 = 42.$$

Once more, we use $y(1.5) = 42$ to approximate

$$y(2) \approx y(1.5) + y'(1.5) \cdot 0.5 = y(1.5) + 2y(1.5)^2 \cdot 0.5 = 42 + 2 \cdot 42^2 \cdot 0.5 = 1806.$$

Then we can summarize these findings in the table

t	0	0.5	1	1.5	2
y	1	2	6	42	1806

Warning: If we solve the above IVP using the method of separation of variables, we would find the exact solution is $y(t) = (1 - 4t)^{-1}$. However, this solution approaches infinity as $t \to 1/4$ and so actually, the solution does not exist for for $t > 1/4$, so the above approximation with Euler's method does not actually approximate any "true solution". We will study this phenomenon in more detail next week.

7/01 Method of integrating factors

Warmup. First let's look at some differential equations that we already know how to solve.

Warmup 1. Let $f(t)$ and $\mu(t)$ be any functions relying only on t. Then consider the differential equation

$$\frac{d}{dt}(\mu(t)y) = f(t).$$

This differential equation can also be easily solved for $y(t)$. First we solve for $y(t)\mu(t)$ by taking the anti-derivative of both sides. If $F(t)$ is the anti-derivative of $f(t)$, then we find

$$\mu(t)y(t) = F(t) + C.$$

Next we can solve for $y(t)$ by simply dividing both sides by $\mu(t)$ which gives us the general solution

$$y(t) = \frac{1}{\mu(t)}(F(t) + C).$$

Warmup 2. Let $p(t)$ be any function relying only on t and consider the differential equation

$$\mu'(t) = p(t)\mu(t)$$

where now $\mu(t)$ is not known: it is the function we are trying to solve for. We can solve this by separation of variables. First rewrite the differential equation as

$$\frac{1}{\mu}d\mu = p(t)dt.$$

Now we take the anti-derivative of both sides. If $P(t)$ is the anti-derivative of $p(t)$ this yields

$$\begin{aligned} \ln|\mu(t)| &= P(t) + C. \\ \implies \quad |\mu(t)| &= e^{P(t)+C} \\ \implies \quad \mu(t) &= \pm e^{C} e^{P(t)}. \end{aligned}$$

We can redefine C to get the equivalent general solution

$$\mu(t) = Ce^{P(t)}.$$

The method of integrating factors. Today we will learn a method that will allow us to solve the following differential equations (along with many more):

(a) $y' + y = t$.
(b) $y' + \frac{1}{t}y = \frac{e^t}{t}$.
(c) $y' + \frac{1}{1+t}y = 2$.
(d) $y' + \frac{1}{2}y = 10$.

Notice all of the differential equations above are in the form

$$y' + p(t)y = g(t)$$

where $p(t)$ and $g(t)$ are some known functions depending only on t and we must solve for $y(t)$. We currently have no method to solve for y in a differential equation of this form. There is a trick, however, to simplify this differential equation into a simpler form by multiplying both sides by a carefully chosen function $\mu(t)$. Before choosing $\mu(t)$, let's multiply both sides differential equation by $\mu(t)$ to see what choice might be useful:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t). \tag{47}$$

Notice the left side of this differential equation is close to $\frac{d}{dt}(\mu(t)y)$ by the product rule:

$$\frac{d}{dt}(\mu(t)y) = \mu(t)y' + \mu(t)p(t)y.$$

Therefore, if we choose $\mu(t)$ so that

$$\mu'(t) = \mu(t)p(t) \tag{48}$$

then the left side of (47) is *exactly* $\frac{d}{dt}(\mu(t)y)$ and so we can rewrite the differential equation as

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t).$$

This differential equation is structurally the same as Warmup 1 above and so we should be able to solve for $y(t)$. Note that we have to find $\mu(t)$ that satisfies (48), but this is the same problem as Warmup 2 above and so this is also doable.

Next we will restructure the discussion above into a sequence of steps that should allow us to solve equations of the form

$$y' + p(t)y = g(t) \tag{49}$$

The following method is called ***the method of integrating factors***.

Step 1. Multiply both sides of (49) by $\mu(t)$ which we'll determine later

$$\mu(t)y'(t) + \mu(t)p(t)y = g(t)\mu(t)$$

Step 2. We pick $\mu(t)$ that solves

$$\mu'(t) = p(t)\mu(t).$$

Step 3. Now we can rewrite this differential equation as

$$\frac{d}{dt}(\mu(t)y) = g(t)\mu(t)$$

because our choice of $\mu(t)$ implies

$$\frac{d}{dt}(\mu(t)y) = \mu(t)y' + \mu'(t)y = \mu(t)y' + p(t)\mu(t)y = g(t)\mu(t).$$

Step 4. Solve the easier differential equation

$$\frac{d}{dt}(\mu(t)y) = g(t)\mu(t).$$

Examples. Now let's use this method to solve some concrete differential equations.

Problem. Solve the differential equation

$$y' + y = t. \tag{50}$$

Solution. We follow the steps in the method of integrating factors. Step 1 is multiply through by some $\mu(t)$ which we'll determine later:

$$\mu(t)y' + \mu(t)y = \mu(t)t.$$

For Step 2, we need to find some $\mu(t)$ so that $\mu'(t) = \mu(t)$. We know from our previous studies that $\mu(t) = e^t$ works for this! With this choice of $\mu(t)$, we can apply Step 3 and rewrite the differential equation in the simpler form

$$\frac{d}{dt}(\mu(t)y) = \mu(t)y' + \mu'(t)y = \mu(t)y' + \mu(t)y = \mu(t)t.$$

Thus all that remains is Step 4: to solve this simpler differential equation

$$\frac{d}{dt}(e^t y) = te^t$$

where we have substituted in $\mu(t) = e^t$. To solve this differential equation, we take the anti-derivative of both sides. We can compute anti-derivative of the right side with integration by parts:

$$\int te^t dt = te^t - \int e^t dt = te^t - e^t + C.$$

The anti-derivative of the left side is just $e^t y$ and so we have the implicit general solution

$$e^t y = te^t - e^t + C.$$

We can solve for y explicitly by multiplying both sides by e^{-t} which yields the general solution

$$y = t - 1 + Ce^{-t}. \tag{51}$$

Let's check this is a solution to (50). Indeed, we can compute

$$y' + y = (1 - Ce^t) + (t - 1 + Ce^{-t}) = t$$

and so this works!

Problem. Solve the IVP

$$\begin{cases} ty' + y = e^t \\ y(1) = e. \end{cases} \tag{52}$$

Some Terminology. First notice the above differential equation is not in the form

$$y' + p(t)y = g(t) \tag{53}$$

that we need to apply the method of integrating factors. However, we can rewrite (52) in this form by dividing both sides by t. This yields

$$y' + \frac{1}{t}y' = \frac{1}{t}e^t. \tag{54}$$

We call a first order differential equation that can be written in this form $y' + p(t)y = g(t)$ for some functions $p(t)$ and $g(t)$ ***linear***.

Solution. Now we continue to solve (54) using the method of integrating factors. Following Step 1, we multiply by some $\mu(t)$, which gives

$$\mu(t)y' + \frac{1}{t}\mu(t)y = \frac{1}{t}\mu(t)e^t. \tag{55}$$

Now we see that we require

$$\mu'(t) = \frac{1}{t}\mu(t)$$

We can solve for such a $\mu(t)$ using separation of variables. Indeed,

$$\begin{aligned} & \mu'(t) = \frac{1}{t}\mu(t) \\ \implies & \frac{d\mu}{dt} = \frac{1}{t}\mu(t) \\ \implies & \frac{1}{\mu}d\mu = \frac{1}{t}dt \\ \implies & \int \frac{1}{\mu}d\mu = \int \frac{1}{t}dt \\ \implies & \ln|\mu| = \ln|t| + C \\ \implies & \mu(t) = Ct. \end{aligned}$$

Thus the simplest choice that works is $\mu(t) = t$. Therefore, making this substitution for μ, we can apply Step 3 to rewrite (55) as

$$\frac{d}{dt}(ty) = e^t.$$

Taking the anti-derivative of both sides, we find

$$ty = e^t + C.$$

Thus we have the general solution

$$y(t) = \frac{e^t + C}{t}.$$

Let's check this is a solution by plugging it into the original differential equation (52). The left side becomes

$$t\frac{d}{dt}\left(\frac{e^t+C}{t}\right)+\frac{e^t+C}{t}=t\left(\frac{te^t-(e^t+C)}{t}\right)+\frac{e^c+C}{t}=e^t$$

which agrees with the right side.

To find the particular solution, we use the initial value $y(1) = e$. Plugging this in to the general solution yields

$$e=\frac{e^1+C}{1} \implies C=0.$$

Therefore $C = 0$ and so our particular solution is

$$y(t)=\frac{1}{t}e^t$$

7/03 Applications of integrating factors

A mixing problem. Consider a container initially holding 1 L of fresh water. Water is pumped out at 1 L/min and salt water with a concentration of 1 g/L is pumped in at a rate of 2 L/min. How does the mass $m(t)$ of salt change over time?

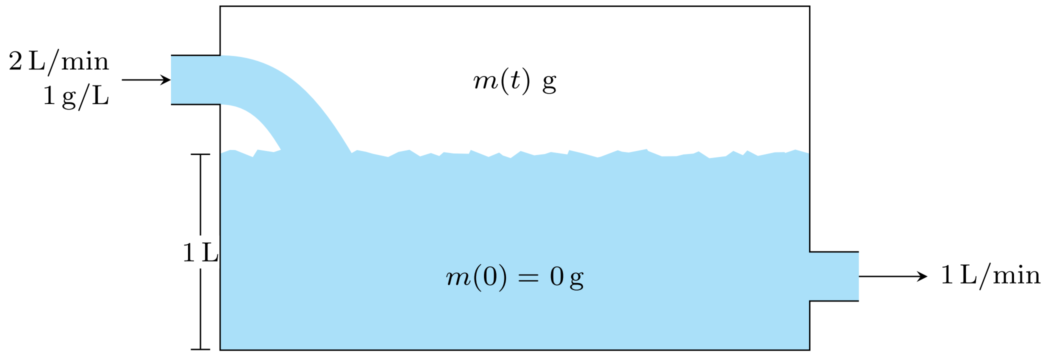


Figure 9. Mixing problem with changing volume

Solution. Recall the rate of change of the salt always satisfies

$$\frac{dm}{dt} = \text{rate mass comes in} - \text{rate mass goes out}.$$

Furthermore, we know

$$\text{rate mass comes in} = \text{input concentration} \cdot \text{input flow rate} = 1\,\text{g/L} \cdot 2\,\text{L/min} = 2\,\text{g/min}.$$

Note this rate is in grams per minute as expected. It is a useful strategy to examine the units as a way to check that an equation or differential equation makes sense. Recall that when computing the rate mass goes out, we use the volume of water in the container. However, in this example, the volume of water is constantly changing because the rate water comes into the container is not the same as the rate water leaves the container. We can compute how the volume changes by

$$\text{volume} = \text{initial volume} + \text{rate volume changes} \cdot \text{change in time} = 1 + (2-1) \cdot t = 1 + t.$$

With this, we can compute

$$\text{rate mass goes out} = \frac{\text{mass}}{\text{volume}} \cdot \text{rate mass goes out} = \frac{m(t)}{1+t} \cdot 1$$

To summarize,

$$\frac{dm}{dt} = \text{rate mass comes in} - \text{rate mass goes out} = 2 - \frac{m(t)}{1+t}.$$

By combining the above differential equation with the initial condition $m(0) = 0$ (because the tank starts with fresh water), we have reduced this problem to the IVP

$$\begin{cases} \frac{dm}{dt} = 2 - \frac{m}{1+t} \\ m(0) = 0. \end{cases} \tag{56}$$

As usual, we begin by finding the general solution to the differential equation. We can rewrite the differential equation in the following form, ready to apply integrating factors.

$$m' + \frac{1}{1+t} m = 2.$$

First we multiply by some to be determined $\mu(t)$:

$$\mu(t) m' + \frac{m(t)}{1+t} m = 2\mu(t). \tag{57}$$

Second, we find some $\mu(t)$ that satisfies

$$\frac{dm}{dt} = \frac{m}{1+t}. \tag{58}$$

Notice the function $\mu(t) = 1 + t$ is such a $\mu(t)$, which we can find using separation of variables. Thus we can rewrite (57) as

$$\frac{d}{dt}((1+t)m) = 2(1+t).$$

Taking the anti-derivative of both sides leads us to

$$(1+t)m = 2t + t^2 + C.$$

Therefore we have the general solution

$$m(t) = \frac{2t + t^2 + C}{1+t}.$$

Now use $m(0) = 0$ to get

$$0 = \frac{0 + 0 + C}{1 + 0} \implies C = 0.$$

Therefore we have the particular solution

$$m(t) = \frac{t(t+2)}{t+1}.$$

Drag. Consider a parachuter descending at velocity $v(t)$. The gravitational pull of the earth increases this velocity at a rate g (about 10 m/s^2). If this were the only factor, the velocity of the parachuter would keep increasing and reach dangerous speeds. This is not what happens, though: there is a drag force that pushes back up on the parachute. Furthermore, this drag force increases as the velocity of the parachute increases. Suppose that this drag is proportional to velocity. In general, a falling object satisfies the differential equation

$$\frac{dv}{dt} = g - \alpha v \tag{59}$$

where α is the drag constant that depends on the object. For a feather, α would be very large as there is a lot of drag, but for a bowling ball α will be very small.

Problem. A basketball is dropped off a cliff with initial velocity 0. Suppose the drag constant is $\alpha = \frac{1}{2}\mathrm{s}^{-1}$ and $g = 10\,\mathrm{m/s}^2$. How does the velocity $v(t)$ of the basketball change over time?

Solution. The velocity of the basketball satisfies the initial value problem

$$\begin{cases} \frac{dv}{dt} = 10 - \frac{v}{2} \\ v(0) = 0. \end{cases} \tag{60}$$

We can solve this differential equations with either separation of variables or integrating factors. We will use integrating factors for the practice. First rewrite the differential equation as

$$v' + \frac{1}{2}v = 10.$$

First multiply by some $\mu(t)$ to be determined later.

$$\mu(t)v' + \frac{\mu(t)}{2}v = 10\mu(t) \tag{61}$$

We hope to find $\mu(t)$ so that

$$\frac{d\mu}{dt} = \frac{1}{2}\mu(t),$$

and $\mu(t) = e^{t/2}$ works. Thus we can rewrite (61) as

$$\frac{d}{dt}(e^{t/2}v) = 10e^{t/2}.$$

To solve for v we take the anti-derivative to get

$$e^{t/2}v = 20e^{t/2} + C$$

and so we find the general solution

$$v(t) = 20 + Ce^{-t/2}.$$

Now use the initial condition $v(0) = 0$ to conclude

$$0 = 20 + C \implies C = -20.$$

Therefore we conclude

$$v(t) = 20(1 - e^{-t/2}).$$

Additional question: terminal velocity. The ***terminal velocity*** of a falling object with velocity $v(t)$ is the velocity this object approaches; that is, the terminal velocity is $\lim_{t\to\infty} v(t)$. What is the terminal velocity of the basketball in the previous question?

To answer this question, we compute

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 20(1 - e^{-t/2}) = 20.$$

Thus we calculate the terminal velocity of the basketball to be 20 m/s. Alternatively, note the terminal velocity will be an equilibrium solution: an object falling at its terminal velocity will stay at precisely this velocity. Therefore, we could have simply studied the equilibrium solutions of the original differential equation

$$\frac{dv}{dt} = 10 - \frac{v}{2}.$$

Setting $\frac{dv}{dt} = 0$, an equilibrium solution v must satisfy

$$0 = 10 - \frac{v}{2} \implies v = 20.$$

Thus this allows us to conclude $v = 20$ is an equilibrium solution without solving the differential equation.

Additional question: position over time. Let $x(t)$ be how far the ball is down the cliff at time t. Solve for the function $x(t)$.

Recall that the derivative of position $x(t)$ is the velocity $v(t)$. That is, $x(t)$ satisfies the differential equation $x'(t) = v(t)$. Furthermore, we know $x(0) = 0$ and so we are looking to solve the initial value problem

$$\begin{cases} x'(t) = v(t) \\ x(0) = 0. \end{cases}$$

To solve the differential equation $x'(t) = v(t)$ we simply need to take the anti-derivative of both sides. That is, the general solution is

$$x(t) = \int v(t)dt = \int 20(1 - e^{-t/2})dt = 20t + 40e^{-t/2} + C.$$

Then by $x(0) = 0$ we find

$$0 = 40 + C \implies C = -40.$$

Thus the particular solution is

$$x(t) = 20t + 40(e^{-t/2} - 1).$$

Population Models. Consider a salmon population with constant of proportionality $r = 2$ years^{-1} and salmon is harvested at a rate of $2t$ salmon/year where t is time in years, and at $t = 0$, there are 10 salmon. Assuming a simple exponential model with harvesting, how does the salmon population $P(t)$ evolve over time?

Solution. We model the change in population of the salmon by

$$\frac{dP}{dt} = 2P - 2t.$$

This together with the initial condition $P(0) = 10$ gives us an initial value problem. Preparing to apply integrating factors, rewrite the differential equation as

$$P' - 2P = -2t.$$

Next multiply the differential equation by some $\mu(t)$ to be determined later.

$$\mu(t)P' - 2\mu(t)P = -2t\mu(t). \tag{62}$$

We want a $\mu(t)$ such that

$$\mu'(t) = -2\mu(t)$$

and so we conclude $\mu(t) = e^{-2t}$ works. Thus we can rewrite (62) as

$$\frac{d}{dt}(e^{-2t}P) = -2te^{-2t}.$$

Now we must use integration by parts to compute the anti-derivative

$$\int -2te^{-2t}dt = te^{-2t} - \int e^{-2t}dt = te^{-2t} + \frac{1}{2}e^{-2t} + C.$$

Therefore we conclude

$$e^{-2t}P = te^{-2t} + \frac{1}{2}e^{-2t} + C,$$

and so

$$P(t) = \frac{1}{2} + t + Ce^{2t}.$$

The initial condition $P(0) = 10$ implies

$$10 = \frac{1}{2} + C \implies C = 19/2.$$

Thus we get the particular solution

$$P(t) = \frac{1}{2} + t + \frac{19}{2}e^{2t}.$$

7/05 Existence and Uniqueness of first order ODEs

Uniqueness of Solutions. Generally when solving an initial value problem, we work under the assumption there is only one solution. However, this is not always the case if the differential equation is not "nice". An example is given below.

Example of two solutions to an IVP. Consider the initial value problem

$$\begin{cases} y' = 2\sqrt{|y|} \\ y(0) = 0. \end{cases}$$

The equilibrium solution $y(t) = 0$ is a solution to this IVP. However, notice that $y(t) = t^2$ is also a solution as $\frac{d}{dt}(t^2) = 2t$ and $2\sqrt{|t^2|} = 2t$. The problem with this differential equation is that $2\sqrt{|y|}$ is not differentiable at $y = 0$. Luckily, if there aren't such problems with the differential equation, solutions will always be unique. These theorems are stated below.

Uniqueness Theorems.

Theorem (Uniqueness theorem for 1st order ODEs)**.** A solution to the initial value problem

$$\begin{cases} y' = f(y) \\ y(t_0) = y_0 \end{cases}$$

is unique if $f(y)$ is continuous and $\frac{d}{dy}f(y)$ is continuous.

We often also encounter differential equations that depend on time t. If you have learned about partial derivatives, we can state the uniqueness theorem in this more general case:

Theorem (Uniqueness theorem for 1st order ODEs)**.** The solution to the initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

is unique if $f(t, y)$ is continuous and $\frac{\partial}{\partial t}f(t, y)$ is continuous (by "continuous", I mean continuous with respect to both t and y).

Another example. Consider the IVP

$$\begin{cases} y' = -\sqrt{1 - y^2} \\ y(0) = 1. \end{cases}$$

Then once again we have the equilibrium solution $y(t) = 1$. However, note $y(t) = \cos t$ is also a solution. Indeed, $\frac{d}{dt}(\cos t) = -\sin t$ and also $-\sqrt{1 - \cos^2 t} = -\sin t$. Thus this IVP must not meet this conditions for the uniqueness theorem. Why?

The most pressing issue is $-\sqrt{1 - y^2}$ is not defined for $y > 1$. Further, note $\frac{d}{dy}(-\sqrt{1 - y^2}) = \frac{y}{\sqrt{1-y^2}}$, so the derivative approaches ∞ as $y \to 1$. Thus it is impossible to extend the function $-\sqrt{1 - y^2}$ so that its derivative is continuous.

It is not always possible to find an explicit solution to an initial value problem. Thus in application, we often must resort to numerically approximating the solution (by using Euler's method for example). However, it only makes sense to numerically approximate *the* solution if we know there is only one solution! This is why the uniqueness theorem above is so important. However, there is a question that is even more important: does *any* solution even exist at all?! We discuss this next.

Existence of Solutions. For existence, even if the differential equation is nice, the solution might not exist for all time. For example, consider the following nice differential equation:

Example of a solution that only exists for some time. Consider the IVP

$$\begin{cases} y' = y^2 \\ y(0) = 1. \end{cases}$$

Solving the differential equation with separation of variables yields the general solution

$$y = \frac{-1}{t+C}$$

and using the initial condition $y(0) = 1$ yields $C = -1$ and so we have the particular solution

$$y = \frac{1}{1-t}.$$

However, note this function is not defined for $t = 1$.

Another example of a solution that only exists for some time. Consider the following IVP, which resembles a logistic population model

$$\begin{cases} y' = y(y-1) \\ y(0) = -1. \end{cases}$$

We found the general solution for the logistic growth model to be (40). In this case, the general solution is

$$y = \frac{Ce^t}{1+Ce^t}.$$

Then the initial condition $y(0) = -1$ implies $C = -1/2$ and so the particular solution is

$$y = \frac{-e^t}{2-e^t}.$$

However, this function is not defined for $t = \ln(2)$.

In the above examples, there exists a unique solution, but this solution blows up asymptotically so that the solution is only defined for a small amount of time. However, the situation could be much worse: if the differential equation is not nice, it is possible for there to be no solution at all as in the following example.

Example of IVP with no solution. Consider the initial value problem

$$\begin{cases} y' = 1, & y \le 0 \\ y' = -1, & y > 0 \\ y(0) = 0. \end{cases}$$

Any solution to this differential equation would be forced to bounce back and forth between the upper half plane and the lower half plane in an impossible way! In fact, we could formally prove that no solution exists even for a short amount of time by using the mean value theorem.

The problem is that function $f(y)$ defining the differential equation by $y' = f(y)$ is not continuous! Luckily, so long as this function is continuous we will avoid this situation and have a solution – at least for a short amount of time.

Theorem (Existence theorem for 1st order ODEs)**.** There exists a solution to the initial value problem

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

so long as $f(t,y)$ is continuous near (t_0, y_0).

7/12 Introduction to linear 2nd order ODEs

Recall a 2nd order ordinary differential equation is a differential equation that involves the second derivative of the unknown function (but no higher derivatives).

First examples of 2nd order ODEs.

First example of 2nd order ODE. Find the general solution to

$$y'' - \alpha y' = 0. \tag{63}$$

To solve this differential equation, we can first find y'. If we regard y' as the unknown function, this becomes a first order differential equation.

$$(y')' = \alpha(y').$$

We have seen previously that this differential equation has general solution

$$y' = Ce^{\alpha t}.$$

Next we can solve for y by taking the anti-derivative of both sides of the above:

$$y(t) = \frac{C}{\alpha}e^{\alpha t} + C_2.$$

Letting $C_1 = C/\alpha$, we get the general solution

$$y(t) = C_1 e^{\alpha t} + C_2. \tag{64}$$

Second example of 2nd order ODE. Find as many solutions to $y'' - y = 0$ as you can.

Note this differential equation is equivalent to $y'' = y$, so we are looking for functions whose 2nd derivative is the function itself. After some trial and error, we can guess that e^t is a solution. Furthermore, e^{-t} also works as a solution. In fact, we can check the linear combination $y(t) = C_1 e^t + C_2 e^{-t}$ for any constants C_1, C_2 will be a solution:

$$y''(t) = \frac{d^2}{dt^2}(C_1 e^t + C_2 e^{-1}) - \frac{d}{dt}(C_1 e^t - C_2 e^{-t}) = C_1 e^t + C_2 e^{-t} = y(t).$$

It turns out that $y(t) = C_1 e^t + C_2 e^{-t}$ contains all possible solutions, so this is the general solution. We will study why this encapsulates all possible solutions later.

We can extend the technique to the following ODE.

Third example of 2nd order ODE. Find as many solutions to the following differential equation as possible:

$$y'' + y' - 2y = 0.$$

Once again, we will begin by guessing solutions. From the above examples, we see that solutions to 2nd order equations tend to be of the form $e^{\lambda t}$ for some λ. Therefore, we plug $e^{\lambda t}$ into the above equation and solve for which λ will make this a solution:

$$\begin{aligned}
& \frac{d^2}{dt^2}e^{\lambda t} + \frac{d}{dt}e^{\lambda t} - 2e^{\lambda t} = 0 \\
\iff \quad & \lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 2e^{\lambda t} = 0 \\
\iff \quad & e^{\lambda t}(\lambda^2 + \lambda - 2) = 0 \\
\iff \quad & \lambda^2 + \lambda - 2 = 0 \\
\iff \quad & (\lambda + 2)(\lambda - 1) = 0 \\
\iff \quad & \lambda = -2 \quad \text{or} \quad \lambda = 1.
\end{aligned}$$

Thus we conclude e^t and e^{-2t} are the two solutions of this form. But also, we can check that $y(t) = C_1e^t + C_2e^{-2t}$ is a solution for any choice of C_1 and C_2:

$$\begin{aligned}
&\frac{d^2}{dt^2}(C_1e^t + C_2e^{-2t}) + \frac{d}{dt}(C_1e^t + C_2e^{-2t}) - 2(C_1e^t + C_2e^{-2t}) \\
&= (C_1e^t + 4C_2e^{-2t}) + (C_1e^t - 2C_2e^{-2t}) + (-2C_1e^t - 2C_2e^{-2t}) \\
&= C_1e^t(1 + 1 - 2) + C_2e^t(4 - 2 - 2) = 0.
\end{aligned}$$

It turns out $y(t) = C_1e^t + C_2e^{-2t}$ encapsulates all possible solutions and so is the general solution.

Some terminology. It turns out this technique of guessing 2 solutions, then considering all linear combinations always works for differential equations of the form

$$y'' + by' + cy = 0 \tag{65}$$

where b and c are constants. Before elaborating on this, we first introduce some terminology: a 2nd order differential equation is called ***linear*** if it is of the form

$$y'' + p(t)y' + q(t)y = g(t). \tag{66}$$

for some functions $p(t), q(t), g(t)$ depending only on t. Furthermore, a linear 2nd order differential equation is called ***homogeneous*** if it is of the form

$$y'' + p(t)y' + q(t)y = 0. \tag{67}$$

The differential equations we will study are a special case of this. We say a homogeneous linear 2nd order differential equation ***has constant coefficients*** if it is of the form

$$y'' + by' + cy = 0. \tag{68}$$

Homogeneous linear differential equations (which includes those with constant coefficients) have the following ***superposition principle***: if $y_1(t)$ and $y_2(t)$ are solutions to (67), then the linear combination $y(t) = C_1y_1(t) + C_2y_2(t)$ is also a solution to (67). This is because

$$\begin{aligned}
y'' + p(t)y' + q(t)y &= \frac{d^2}{dt^2}(C_1y_1 + C_2y_2) + p(t)\frac{d}{dt}(C_1y_1 + C_2y_2) + q(t)(C_1y_1 + C_2y_2) \\
&= (C_1y_1'' + C_2y_2'') + p(t)(C_1y_1' + C_2y_2') + q(t)(C_1y_1 + C_2y_2) \\
&= C_1(y_1'' + p(t)y_1' + q(t)y_1) + C_2(y_2'' + p(t)y_2' + q(t)y_2) = 0.
\end{aligned}$$

The case of two real roots. Consider a general 2nd order linear homogeneous differential equation with constant coefficients:

$$y'' + by' + c = 0. \tag{69}$$

When encountering a differential equation of this type, we will always first find solutions of the form $e^{\lambda t}$ for some λ. Note having such a solution is equivalent to

$$\begin{aligned}
&\frac{d^2}{dt^2}e^{\lambda t} + b\frac{d}{dt}e^{\lambda t} + ce^{\lambda t} = 0 \\
\iff \quad & e^{\lambda t}(\lambda^2 + b\lambda + c) = 0 \\
\iff \quad & \lambda^2 + b\lambda + c = 0.
\end{aligned}$$

Thus the problem always reduces to studying roots of the polynomial $\lambda^2 + b\lambda + c = 0$, which is called the ***characteristic polynomial***. However, $\lambda^2 + b\lambda + c = 0$ does not always have a real solution. We will start by studying the case when $\lambda^2 + b\lambda + c = 0$ has two real solutions $\lambda_1 \neq \lambda_2$. In this case, $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are both solutions to the differential equation (69) and so by the superposition principle

$$y(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} \tag{70}$$

is a solution for any C_1, C_2. In this particular case, (70) is in fact the general solution! That, is every solution to (69) is of the form (70) (we will justify this later).

Problem. Find the general solution to

$$\ddot{y} = 2\dot{y} - 8y = 0 \tag{71}$$

where we are using the notation $\dot{y}(t) = y'(t)$ and $\ddot{y}(t) = y''(t)$.

Solution. We first find solutions of the form $e^{\lambda t}$. Such a solution must satisfy

$$\begin{aligned} & \frac{d^2}{dt^2}e^{\lambda t} - 2\frac{d}{dt}e^{\lambda t} - 8e^{\lambda t} = 0 \\ \iff \quad & e^{\lambda t}(\lambda^2 - 2\lambda - 8) = 0 \\ \iff \quad & (\lambda - 4)(\lambda + 2) = 0 \\ \iff \quad & \lambda = 4 \quad \text{or} \quad \lambda = -2. \end{aligned}$$

Thus the two solutions of this form are e^{4t} and e^{-2t} and so the general solution is

$$y(t) = C_1e^{4t} + C_2e^{-2t}.$$

Problem. Find the particular solution to (71) that satisfies the initial conditions $y(0) = -1$ and $\dot{y}(0) = 8$.

Solution. We found the general solution is

$$y(t) = C_1e^{4t} + C_2e^{-2t}.$$

Now we use the initial conditions to find the constants. Note the initial condition $y(0) = -1$ implies

$$C_1e^{4\cdot 0} + C_2e^{-2\cdot 0} = 8 \implies C_1 + C_2 = -1.$$

Next we compute the derivative of the general solution to be

$$\dot{y}(t) = 4C_1e^{4t} - 2C_2e^{-2t}$$

and thus the initial condition $\dot{y}(0) = 8$ implies

$$4C_1e^{4\cdot 0} - 2C_2e^{-2\cdot 0} = 8 \implies 4C_1 - 2C_2 = 8.$$

Combining this information we have the system of equations

$$\begin{cases} C_1 + C_2 = -1 \\ 4C_1 - 2C_2 = 8. \end{cases}$$

Now you can solve this system however you would like. For example, multiply the first equation by 2 and add the second to conclude

$$-2 + 8 = 2(C_1 + C_2) + (4C_1 - 2C_2) = 6C_1.$$

Thus $6 = 6C_1$ and so $C_1 = 1$. Then we can solve for $C_2 = -1 - C_1 = -1 - 1 = -2$. Thus the particular solution is

$$e^{4t} - 2e^{-2t} = 0.$$

7/15 The case of repeated roots

A problematic example. Find the general solution to

$$y'' - 2y' + y = 0.$$

Let's try our previous strategy. If $y(t) = e^{\lambda t}$, then we need

$$\begin{aligned} & \frac{d^2}{dt^2}e^{\lambda t} - 2\frac{d}{dt}e^{\lambda t} + e^{\lambda t} = 0 \\ \iff \quad & e^{\lambda t}(\lambda^2 - 2\lambda + 1) = 0 \\ \iff \quad & (\lambda - 1)^2 = 0 \\ \iff \quad & \lambda = 1. \end{aligned}$$

Thus we conclude e^t is a solution and by the superposition principle we can conclude Ce^t is a solution for any C. However, we have missed many solutions. Importantly, note te^t is also solution:

$$\begin{aligned} & \frac{d^2}{dt^2}(te^t) - 2\frac{d}{dt}(te^t) + te^t \\ & = \frac{d}{dt}(e^t + te^t) - 2(e^t + te^t) + te^t \\ & = (e^t + e^t + te^t) - 2(e^t + te^t) + te^t = 0. \end{aligned}$$

The problem is the characteristic polynomial $\lambda^2 - 2\lambda + 1$ has 1 root of order 2 as opposed to 2 distinct roots. To see why this is a problem, we need to understand why our technique works when we have 2 distinct roots.

Why our technique works for distinct roots. Consider the differential equation

$$y'' + by' + cy = 0$$

and the corresponding characteristic polynomial equation

$$\lambda^2 + b\lambda + c = 0$$

Suppose this characteristic polynomial has roots λ_1, λ_2. That is, we can write

$$\lambda^2 + b\lambda + c = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

In fact, observe the above implies $b = -(\lambda_1 + \lambda_2)$ and $c = \lambda_1\lambda_2$. Now we can rewrite our differential equation using these roots, which will help us link the behavior of the differential equation to the roots.

$$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0.$$

Next we will use the clever substitution $z(t) = e^{-\lambda_1 t}y(t)$, which means $y(t) = e^{\lambda_1 t}z(t)$. This substitution will drastically simplify our differential equation after applying product rule a few times and canceling like terms:

$$\begin{aligned} & y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0 \\ \iff \quad & \frac{d^2}{dt^2}(e^{\lambda_1 t}z) - (\lambda_1 + \lambda_2)\frac{d}{dt}(e^{\lambda_1 t}z) + \lambda_1\lambda_2 e^{\lambda_1 t} = 0 \\ \iff \quad & e^{\lambda_1 t}((\lambda_1^2 z + 2\lambda_1 z' + z'') - (\lambda_1 + \lambda_2)(\lambda_1 z + z') + \lambda_1\lambda_2 z = 0 \\ \iff \quad & z'' + (\lambda_1 - \lambda_2)z' = 0 \\ \iff \quad & z'' = (\lambda_2 - \lambda_1)z' \end{aligned}$$

where we used product rule to substitute

$$\frac{d}{dt}(e^{\lambda_1 t}z) = \lambda_1 e^{\lambda_1 t}z + e^{\lambda_1 t}z' = e^{\lambda_1 t}(\lambda_1 z + z')$$

and

$$\frac{d^2}{dt^2}(e^{\lambda_1}z) = \lambda_1^2 e^{\lambda_1 t}z + \lambda_1 e^{\lambda_1 t}z' + \lambda_1 e^{\lambda_1 t}z' + e^{\lambda_1 t}z'' = e^{\lambda_1 t}(\lambda_1^2 z + 2\lambda_1 z' + z'').$$

Thus we have deduced

$$(z')' = (\lambda_2 - \lambda_1)z'. \tag{72}$$

Now we can solve for all z' that solves this by separation of variables. If $\lambda_2 \neq \lambda_1$, then we have seen the general solution is

$$z'(t) = Ce^{(\lambda_2-\lambda_1)t}.$$

Thus the general solution for $z(t)$ is any anti-derivative of the right side, which is

$$\int Ce^{(\lambda_2-\lambda_1)t}dt = \frac{C}{\lambda_2-\lambda_1}e^{(\lambda_2-\lambda_1)t} + C_2.$$

Thus by introducing the new constant C_1 we have

$$z(t) = C_1e^{(\lambda_2-\lambda_1)t} + C_2.$$

Thus we can solve for $y(t)$ by

$$y(t) = e^{\lambda_1 t}z(t) = e^{\lambda_1 t}(C_1e^{(\lambda_2-\lambda_1)t} + C_2) = C_1e^{\lambda_2 t} + C_2e^{\lambda_1 t}.$$

Thus we have proved that any solution must be of the form

$$y(t) = C_1e^{\lambda_2 t} + C_2e^{\lambda_1 t}$$

just as promised!

Repeated roots. The above goes slightly differently if $\lambda_1 = \lambda_2$. In particular, for (72) we get

$$(z')' = 0.$$

Thus we conclude $z' = C_1$ for some constant C_1. To solve for z we take the anti-derivative of both sides. The anti-derivative of C_1 is $C_1t + C_2$ and therefore $z(t) = C_1t + C_2$. Then we find the general solution for $y(t)$ is

$$y(t) = e^{\lambda_1 t}z(t) = C_1te^{\lambda_1 t} + C_2e^{\lambda_1 t}. \tag{73}$$

For an example, let's return to our differential equation $y'' - 2y' + y = 0$ which has characteristic polynomial $\lambda^2 - 2\lambda + 1 = (\lambda-1)^2$ with only $\lambda_1 = 1$ as a root. Then by (73) the general solution to $y'' - 2y' + y = 0$ is

$$y(t) = C_1te^t + C_2e^t.$$

Now we can be sure we are not missing solutions!

Examples.

Example 1. Find the general solution to

$$\frac{1}{2}y'' + 2y' + 2y = 0. \tag{74}$$

First we should rewrite this differential equation in the form

$$y'' + 4y' + 4y = 0.$$

Then the characteristic polynomial equation is

$$\begin{aligned} &\lambda^2 + 4\lambda + 4 = 0 \\ \iff \quad &(\lambda+2)^2 = 0. \end{aligned}$$

This has roots $\lambda_1 = -2$ and so the general solution is

$$C_1e^{-2t} + C_2te^{-2t}.$$

Example 1 Extension. Find the particular solution to (74) that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.

First recall the general solution is

$$y(t) = C_1 te^{-2t} + C_2 e^{-2t}.$$

Thus the initial condition $y(0) = 1$ implies

$$1 = C_1 e^{-2\cdot 0} + C_2 \cdot 0 \cdot e^{-2\cdot 0} \implies C_1 = 1.$$

Next use product rule to compute

$$y'(t) = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 te^{-2t} = (-2C_1 + C_2)e^{-2t} - 2C_2 te^{-2t}$$

Therefore the initial condition $y'(0) = 0$ implies

$$0 = -2C_1 + C_2 \implies 0 = -2 + C_2 \implies C_2 = 2.$$

Therefore $C_1 = 1$ and $C_2 = 2$ yields the particular solution

$$y(t) = e^{-2t} + 2te^{-2t}.$$

7/17 Review of complex numbers

Motivation. Complex numbers are quantities $a + b\sqrt{-1}$ involving the "square root of -1". It turns out this concept is extremely useful for our study of differential equations. In particular, we will see that we must understand complex numbers to understand the differential equations that model a mass on a spring or certain electrical circuits. For example, consider the following differential equation.

Problem to motivate complex numbers. Find the general solution to $y'' + y = 0$.

Let's suppose the solution is of the form $y(t) = e^{\lambda t}$. In this case, we see that we require

$$\begin{aligned} & \frac{d^2}{dt^2}e^{\lambda t} + e^{\lambda t} = 0 \\ \iff \quad & e^{\lambda t}(\lambda^2 + 1) = 0 \\ \iff \quad & \lambda^2 = -1. \end{aligned}$$

There are no real solutions to $\lambda^2 = -1$, but we know there are solutions to the differential equation $y'' + y = 0 \iff y'' = -y$. Indeed, both $\sin t$ and $\cos t$ are solutions and therefore by the superposition principle, $C_1 \sin t + C_2 \cos t$ are solutions. We will see that if we suppose there are solutions to $\lambda^2 = -1$, which we will call $\lambda = \pm\sqrt{-1}$, and if $e^{\sqrt{-1}\cdot t}$ and $e^{-\sqrt{-1}\cdot t}$ makes sense, then these are solutions because, for example,

$$\frac{d^2}{dt^2}(e^{\sqrt{-1}\cdot t}) + e^{\sqrt{-1}\cdot t} = (\sqrt{-1})^2 e^{\sqrt{-1}\cdot t} + e^{\sqrt{-1}\cdot t} = 0.$$

and in the same way we can compute $e^{-\sqrt{-1}\cdot t}$ is a solution. We will study what $e^{\sqrt{-1}\cdot t}$ means and how it is related to the solutions $C_1 \sin t + C_2 \cos t$. This technique will generalize and allow us to solve more complicated differential equations!

Complex numbers. The point is that it is extremely useful to treat $\sqrt{-1}$ as a number, for this will lead to new ways of finding solutions to differential equations. Accepting $\sqrt{-1}$ as a number means we can add, multiply, divide, etc. with other numbers. It is common to denote $\sqrt{-1}$ by i for short, so $i = \sqrt{-1}$. So for example:

- We can add $i + i = 2i$.
- We can multiply i by 4 to get a new number $4i$.
- We can multiply i by any real number b to get bi.
- We can add 3 to i to get a number $3 + i$.
- We can add any real a to bi to get $a + bi$.

Thus if we accept $i = \sqrt{-1}$ as a number that we can apply operations to, we must accept $a + bi$ for any real a, b as a number. A number of this form $a + bi$ is called a ***complex number***. In general, we add two complex numbers by

$$(a + bi) + (\alpha + \beta i) = (a + \alpha) + (b + \beta)i.$$

For example,

$$(-1 + 2i) + (3 - i) = (-1 + 3) + (2 - 1)i = 2 + i.$$

A good way to visualize complex numbers is with the ***complex plane***: to visualize a complex number $x + iy$, we plot in at the point (x, y) on the plane. Note the horizontal axis is the familiar numberline where all real numbers live. We often use z to denote an arbitrary complex number $z = x + iy$.

Furthermore, we can multiply complex numbers by distributing:

$$(a + bi) \cdot (\alpha + \beta i) = a\alpha + a\beta i + b\alpha i + b\beta i^2 = (a\alpha - b\beta) + (a\beta + b\alpha)i$$

where we used $i^2 = -1$ by $i = \sqrt{-1}$. For example,

$$(-1 + 2i) \cdot (3 - i) = (-3 + 2) + (1 + 6)i = -1 + 7i.$$

As a simpler example, $2 \cdot (1 + i) = 2 + 2i$.

Next, let's study what division of complex numbers means. For example, what complex number does $1/(2+i)$ represent? We can compute this will the following technique. First note

$$(2+i)\cdot(2-i) = (4+1) + (2-2)i = 5.$$

Therefore, we can compute

$$\frac{1}{2+i} = \frac{1}{2+i}\cdot 1 = \frac{1}{2+i}\cdot\frac{2-i}{2-i} = \frac{1}{5}(2-i) = \frac{2}{5} - \frac{1}{5}i.$$

In general, because $(x+iy)(x-iy) = x^2+y^2$ we find

$$\frac{1}{x+iy} = \frac{1}{x+iy}\cdot\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}.$$

For computations such as the above, it is useful to define the ***complex conjugate*** of a complex number $z = x+iy$ to be the complex number $\overline{z} = x - iy$. Furthermore, we define the ***modulus*** of a complex number $z = x+iy$ to be $|z| = \sqrt{x^2+y^2}$, which represents the distance from $x+iy$ to 0 in the complex plane. Note that $z\overline{z} = x^2+y^2 = |z|^2$. In general, to compute the quotient w/z we can compute

$$\frac{w}{z} = \frac{w}{z}\cdot\frac{\overline{z}}{\overline{z}} = w\cdot\frac{\overline{z}}{|z|^2}.$$

Polar form of a complex number. Note that any $z = x+iy$ with $|z| = 1$ falls on the unit circle and therefore can be written as $z = \cos\theta + i\sin\theta$ for some angle θ. We define

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{75}$$

to be the function that gives us the complex number that is an angle θ along the unit circle.

$$e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

This function $e^{i\theta}$ is an example of a ***complex valued function***: it outputs a complex number. This is in contrast to a ***real valued function*** which outputs a real number. For another (famous) example, note

$$e^{i\pi} = \cos\pi + i\sin\pi = -1.$$

An important question is: why should we define $e^{i\theta} = \cos\theta + i\sin\theta$? For our purposes in differential equations, note that the derivative of this function behaves in the way we want:

$$\frac{d}{d\theta}e^{i\theta} = \frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = ie^{i\theta}.$$

Furthermore we have $e^{i\cdot 0} = \cos(0) + i\sin(0) = 1$ as we would expect. Thus every complex number $z = x+iy$ with $|z| = 1$ can be written as $z = e^{i\theta}$ for some angle θ. Furthermore, by scaling appropriately with some number r, we can write every complex number $z = x+iy$ as $z = re^{i\theta}$; this is called the ***polar form*** of a complex number. We can convert between the forms $x+iy$ and $re^{i\theta}$ in the same way we convert between Cartesian coordinates (x, y) and polar coordinates (r, θ). Given a complex number $z = re^{i\theta}$ in polar form, it can be rewritten in the standard form by

$$re^{i\theta} = r\cos\theta + ir\sin\theta.$$

Conversely, given a complex number $z = x+iy$, we can rewrite this in polar form as follows. First, compute the angle θ using $\tan(\theta) = y/x$. This means $\theta = \arctan(y/x)$ if $x > 0$ and $\theta = \arctan(y/x) + \pi$ if $x < 0$. Then for the scaling factor we have $r = |z| = \sqrt{x^2+y^2}$. One of the primary benefits of the polar form is that multiplication is easy to compute

$$re^{i\theta}\cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}.$$

In general, we define the exponential of a complex number by

$$e^{x+iy} = e^x\cdot e^{iy} = e^x(\cos y + i\sin y).$$

It is often useful to use the relation $e^{i\theta} = \cos\theta + i\sin\theta$ allows us to write $\cos\theta$ and $\sin\theta$ using only $e^{\pm i\theta}$ as follows.

$$\begin{aligned}\frac{1}{2}(e^{i\theta}+e^{-i\theta}) &= \frac{1}{2}\left((\cos(\theta)+i\sin(\theta))+(\cos(-\theta)+i\sin(-\theta))\right)\\ &= \frac{1}{2}\left((\cos(\theta)+i\sin(\theta))+(\cos(\theta)-i\sin(\theta))\right)=\cos\theta\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{2i}(e^{i\theta}-e^{-i\theta}) &= \frac{1}{2i}\left((\cos(\theta)+i\sin(\theta))-(\cos(-\theta)+i\sin(-\theta))\right)\\ &= \frac{1}{2i}\left((\cos(\theta)+i\sin(\theta))-(\cos(\theta)-i\sin(\theta))\right)=\sin\theta\end{aligned}$$

To summarize,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}. \tag{76}$$

This relationship between the exponential $e^{\pm i\theta}$ and the trigonometric functions $\sin\theta$, $\cos\theta$ is very useful for deriving trigonometric identities. For example, we can express the product of sines as the difference of cosines:

$$\begin{aligned}\sin\alpha\sin\beta &= \frac{1}{2i}(e^{i\alpha}-e^{-i\alpha})\frac{1}{2i}(e^{i\beta}-e^{-i\beta}) = \frac{-1}{4}(e^{i(\alpha+\beta)}+e^{-i(\alpha+\beta)}-e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)})\\ &= \frac{-1}{2}\left(\frac{e^{i(\alpha+\beta)}+e^{-i(\alpha+\beta)}}{2}-\frac{e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}}{2}\right) = \frac{-1}{2}(\cos(\alpha+\beta)-\cos(\alpha-\beta)).\end{aligned}$$

More useful is in fact the reverse identity: expressing the difference $\cos\theta - \cos\phi$ of cosines as the product of sines. Indeed, to use the above, we must find α and β such that

$$\begin{cases}\alpha+\beta=\theta\\ \alpha-\beta=\phi.\end{cases}$$

The solution to this is $\alpha = \frac{\theta+\phi}{2}$ and $\beta = \frac{\theta-\phi}{2}$. Therefore using the above, we conclude

$$\cos(\theta)-\cos(\phi) = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right).$$

For another example, note we can compute

$$A\cos(\theta)+B\sin(\theta) = A\frac{e^{i\theta}+e^{-i\theta}}{2}+B\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2}((A-Bi)e^{i\theta}+(A+Bi)e^{-i\theta}).$$

Now write $A - Bi = re^{i\phi}$ in polar form where we can find r and ϕ explicitly as we practiced above. Then $A + Bi = re^{-i\phi}$ and so

$$A\cos(\theta)+B\sin(\theta) = \frac{1}{2}(re^{i\phi}e^{i\theta}+re^{-i\phi}e^{-i\theta}) = r\frac{1}{2}(e^{i(\theta+\phi)}+e^{-i(\theta+\phi)}) = r\cos(\theta+\phi).$$

For our purposes in differential equations, it is important to note that differentiation of complex valued functions generally works as expected. We simply treat i as a number that factors through differentiation and works with chain rule as any other number would. For example,

$$\begin{aligned}\frac{d}{dt}(3it+1) &= 3i\\ \frac{d}{dt}(e^{2it}) &= 2ie^{it}\\ \frac{d}{dt}(a+ib) &= \frac{d}{dt}a+i\frac{d}{dt}b=0\end{aligned}$$

7/19 The case of two complex roots

In our studies so far, we have been finding all real valued functions that solve a differential equation. However, now that we feel comfortable with complex numbers we can extend further to find all complex valued function that solve a differential equation.

Examples of complex valued solutions.

Example 1. Find all the complex solutions to

$$y' = 2.$$

Solution. That is, what are all complex valued functions with anti-derivative 2? The solution is $y(t) = 2t + (a + ib)$. Or alternatively, $y(t) = 2t + C$ where now C is an arbitrary *complex* number.

Example 2. What do you think all the complex solutions to the following differential equation are?

$$y'' + 3y' + 2y = 0.$$

Solution. Note the characteristic polynomial equation is

$$\begin{aligned} & \lambda^2 + 3\lambda + 2 = 0 \\ \iff \quad & (\lambda + 1)(\lambda + 2) = 0. \end{aligned}$$

Thus the roots are $\lambda_1 = -1$ and $\lambda_1 = -2$. In fact, by the same argument that shows $C_1 e^{-t} + C_2 e^{-2t}$ encompasses all real solutions, we can conclude that *all* complex solutions are of the form

$$y(t) = C_1 e^{-t} + C_2 e^{-2t}$$

where C_1 and C_2 are general *complex* numbers.

Example 3. Find all the complex solutions to the differential equation

$$y'' + y = 0.$$

Solution. Note if $y(t) = e^{\lambda t}$, then we need

$$\begin{aligned} & \frac{d^2}{dt^2} e^{\lambda t} + e^{\lambda t} = 0 \\ \iff \quad & e^{\lambda t}(\lambda^2 + 1) = 0 \\ \iff \quad & \lambda^2 + 1 = 0. \end{aligned}$$

Thus we need $\lambda = i$ or $\lambda = -i$. Indeed, note the complex valued functions e^{it} and e^{-it} are solutions. For example,

$$\frac{d^2}{dt^2} e^{it} + e^{it} = i^2 e^{it} + e^{it} = -e^{it} + e^{it} = 0.$$

In fact, the same reasoning that proves we have all solutions when the roots are real works when the roots are complex to give us the general complex solution

$$y(t) = C_1 e^{it} + C_2 e^{-it} \tag{77}$$

where C_1 and C_2 are complex numbers.

Example 4. Find the solution to

$$y'' + y = 0$$

that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Solution. Note that complex solutions include real solutions. Thus the real solution that satisfies this initial condition is hidden in our general complex solution

$$y(t) = C_1 e^{it} + C_2 e^{-it}$$

and we just need to find it! Indeed, plugging the initial condition $y(0) = 1$ into the complex general solution yields

$$C_1 + C_2 = 1.$$

Next compute the derivative

$$y'(t) = iC_1 e^{it} - iC_2 e^{-it}.$$

Thus the initial condition $y'(0) = 0$ implies

$$0 = iC_1 - iC_2 \implies C_1 - C_2 = 0.$$

That is, $C_1 = C_2$. This combined with $C_1 + C_2 = 1$ implies $C_1 = C_2 = \frac{1}{2}$. Thus our solution is

$$y(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \cos(t).$$

While the general complex solution is completely serviceable in solving initial value problems, it is usually more convenient to first find the general *real* solution, then solve the initial value problem. That is, we need to find all choices of C_1 and C_2 that result in a real solution.

General strategy. Consider the differential equation

$$y'' + by' + cy = 0.$$

Then $e^{\lambda t}$ is a solution exactly when λ solves the corresponding characteristic polynomial equation

$$\lambda^2 + b\lambda + c = 0$$

which has solutions

$$\lambda = \frac{-b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}.$$

We are studying the case in which we have complex solutions, meaning $\left(\frac{b}{2}\right)^2 < c$. Note that if this is the case, then by the quadratic equation the two complex solutions will always come in complex conjugate pairs

$$\lambda_1 = \rho + i\omega, \qquad \lambda_2 = \rho - i\omega.$$

Thus we have solutions

$$e^{(\rho+i\omega)t} \qquad \text{and} \qquad e^{(\rho-i\omega)t}.$$

By the same reasoning we used in the case of real roots, we can argue that the general complex solution is

$$y(t) = C_1 e^{(\rho+i\omega)t} + C_2 e^{(\rho-i\omega)t}.$$

Once again, this contains all real solutions: we just need to find them! For example, to get one real solution, we can choose $C_1 = 1/2$ and $C_2 = 1/2$ to find

$$\frac{1}{2}e^{(\rho+i\omega)t} + \frac{1}{2}e^{(\rho-i\omega)t} = e^{\rho t}\left(\frac{1}{2}e^{i\omega t} + \frac{1}{2}e^{-\omega t}\right) = e^{\rho t}\cos(\omega t).$$

Similarly, we can choose $C_1 = 1/(2i)$ and $C_2 = -1/(2i)$ to get

$$\frac{1}{2i}e^{(\rho+i\omega)t} - \frac{1}{2i}e^{(\rho-i\omega)t} = e^{\rho t}\left(\frac{1}{2i}e^{i\omega t} - \frac{1}{2i}e^{-\omega t}\right) = e^{\rho t}\sin(\omega t).$$

From these two real solutions, we can generate many more by the superposition principle:

$$y(t) = C_1 e^{\rho t}\cos(\omega t) + C_2 e^{\rho t}\sin(\omega t).$$

In fact, if you systematically solve for all complex constants C_1 and C_2 that result in a real solution (which is a good exercise), and you will find that the solution above accounts for all possible real solutions and so is the general real solution.

Example 5. Find the general (real) solution to

$$y'' - 2y' + 5y = 0. \tag{78}$$

Solution. Consider the characteristic polynomial equation

$$\begin{aligned} & \lambda^2 - 2\lambda + 5 = 0 \\ \iff \quad & (\lambda - 1)^2 + 4 = 0 \\ \iff \quad & \lambda - 1 = \pm\sqrt{-4} \\ \iff \quad & \lambda = 1 \pm 2i. \end{aligned}$$

Thus by our reasoning above, the general real solution will be given by

$$y(t) = e^t(C_1 \cos(2t) + C_2 \sin(2t)).$$

Example 5 Extension. Find the particular solution to (78) satisfying the initial conditions $y(0) = 1$ and $y'(0) = -1$.

Solution. Plugging in the initial condition $y(0) = 1$ into our general real solution, we find

$$1 = e^0(C_1 \cos(2 \cdot 0) + C_2 \sin(2 \cdot 0)) \implies C_1 = 1.$$

Next compute the derivative

$$y'(t) = e^t(C_1 \cos(2t) + C_2 \sin(2t)) + e^t(-2C_1 \sin(2t) + 2C_2 \cos(2t)).$$

Next plug in the initial condition $y'(0) = -1$ to get

$$-1 = e^0(C_1 \cos(2 \cdot 0) + C_2 \sin(2 \cdot 0)) + e^0(-2C_1 \sin(2 \cdot 0) + 2C_2 \cos(2 \cdot 0)) \implies C_1 + 2C_2 = -1.$$

Using $C_1 = 1$ this yields $1 + 2C_2 = -1 \implies C_2 = -1$. Therefore, the particular solution is

$$y(t) = e^t(\cos(2t) - \sin(2t))$$

In fact, we can use the identity

$$A\cos(\theta) + B\sin(\theta) = r\cos(\theta + \phi)$$

where $A - iB = re^{i\phi}$. In our case, consider $A = 1$, $B = -1$, and $\theta = 2t$. Thus because $1 + i = \sqrt{2}e^{i\pi/4}$ we find

$$\cos(2t) - \sin(2t) = \sqrt{2}\cos(2t + \pi/4).$$

Therefore we can rewrite our solution as

$$y(t) = \sqrt{2} \cdot e^t \cos(2t + \pi/4).$$

Summary of techniques. In general, consider the differential equation

$$y'' + by' + cy = 0$$

with characteristic polynomial equation

$$\lambda^2 + b\lambda + c = 0.$$

There are three different forms the solutions to this differential equation can take:

(1) Two distinct real solutions $\lambda_1 \neq \lambda_2$. In this case, the general solution is
$$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$
(2) One real solution λ_1. In this case, the general solution is
$$C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}.$$
(3) Two complex solutions $\rho \pm i\omega$. In this case, the general solution is
$$e^{\rho t}(C_1 \cos(\omega t) + C_2 \sin(\omega t)).$$

7/22 The non-homogeneous case

Motivating Example. Consider the differential equation

$$y'' + y = 2e^t$$

and try to find some solutions $y(t)$. For example, one solution is $y(t) = e^t$ because

$$\frac{d^2}{dt^2}e^t + e^t = e^t + e^t = 2e^t.$$

Are there any more solutions? One other solution is actually $e^t + \cos t$ because

$$\frac{d^2}{dt^2}(e^t + \cos t) + (e^t + \cos t) = (e^t - \cos t) + (e^t + \cos t) = 2e^t.$$

Notice how the $\cos t$ conveniently cancels out. In fact, if $z(t)$ solves $z'' + z = 0$, then $z(t) + e^t$ is a solution to $y'' + y = 2e^t$ because

$$\frac{d^2}{dt^2}(z + e^t) + (z + e^t) = z'' + z + 2e^t = 2e^t.$$

Therefore one way to generate many solutions is to find the general solution to $z'' + z = 0$ which we found last time to be $C_1 \cos t + C_2 \sin t$ and then

$$C_1 \cos t + C_2 \sin t + e^t$$

will be a solution for any C_1 and C_2. In fact, this will be the general solution and we will see why next.

Solving non-homogeneous differential equations. We will now study differential equations of the form

$$y'' + by' + cy = f(t), \tag{79}$$

which is linear and has constant coefficients, but is non-homogeneous. The strategy is that if we can just find one *particular solution* y_p to this differential equation then we can generate many more solutions. If $y_h(t)$ is the general solution to the corresponding homogeneous differential equation

$$y'' + by' + cy = 0, \tag{80}$$

then $z(t) + y_p(t)$ solves (79) because

$$\begin{aligned}\frac{d^2}{dt^2}(y_h + y_p) + b\frac{d}{dt}(y_h + y_p) + c(y_h + y_p) &= (y_h'' + y_p'') + b(y_h' + y_p') + c(y_h + y_p) \\ &= (y_h'' + by_h' + y_h) + (y_p'' + by_p' + cy_p) = 0 + f(t) = f(t).\end{aligned}$$

In fact, $y_h(t) + y_p(t)$ is the general solution to (79). The reason for this is that if $y(t)$ is any solution to (79), then note $y(t) - y_p(t)$ must be a solution to the corresponding homogeneous differential equation (80) because

$$\frac{d^2}{dt^2}(y - y_p) + b\frac{d}{dt}(y - y_p) + c(y - y_p) = (y'' + by' + cy) - (y_p'' + by_p' + cy_p) = f(t) - f(t) = 0.$$

Therefore $y(t) - y_p(t) = y_h(t)$ where $y_h(t)$ is some solution to the homogeneous differential equation, and thus $y(t)$ must be of the form $y(t) = y_h(t) + y_p(t)$.

To summarize, to find the general solution to

$$y'' + by' + cy = f(t),$$

Step 1. Find a particular solution y_p so that $y_p'' + by_p' + cy_p = f(t)$. This often will require some educated guessing.

Step 2. Find the general solution y_h to the corresponding homogeneous differential equation $y'' + by' + cy = 0$.

Step 3. Then the general solution is

$$y(t) = y_h(t) + y_p(t).$$

Examples.

Example 1. Find the general solution to

$$y'' + 2y' + y = 2e^{-2t}.$$

Solution. First we must find a particular solution, which will require an educated guess. Because there is an e^{-2t} on the right, a good guess is that the particular solution will be of the form Ce^{-2t} for some particular choice of C. To find which C works we plug in

$$\begin{aligned} & \frac{d^2}{dt^2}(Ce^{-2t}) + 2\frac{d}{dt}(Ce^{-2t}) + (Ce^{-2t}) = 2e^{-2t} \\ \iff \quad & 4Ce^{-2t} - 4Ce^{-2t} + Ce^{-2t} = 2e^{-2t} \\ \iff \quad & Ce^{-2t} = 2e^{-2t} \\ \iff \quad & C = 2. \end{aligned}$$

That is, $y_p(t) = 2e^{-2t}$ is a particular solution. Next we need to find the general solution to the corresponding homogeneous equation

$$y'' + 2y' + y = 0.$$

Note the characteristic polynomial equation is

$$\lambda^2 + 2\lambda + 1 = 0 \implies \quad (\lambda + 1)^2 = 0 \implies \lambda = -1.$$

Thus here we have a repeated root $\lambda = -1$ and so the general solution to the homogeneous equation is

$$y_h(t) = C_1 e^{-t} + C_2 te^{-t}.$$

Combining the homogeneous and particular solutions, we arrive at the general solution

$$y(t) = C_1 e^{-t} + C_2 te^{-t} + 2e^{-2t}.$$

Example 2. Find the general solution to

$$y'' + 3y' + 2y = 14\cos(2t) + 2\sin(2t). \tag{81}$$

Solution. Again, we must first find a particular solution. We guess that there is a particular solution of the form $A\cos(2t) + B\sin(2t)$ for some choice of A and B. We plug this into the differential equation to determine A and B.

$$\begin{aligned} & \frac{d^2}{dt^2}(A\cos(2t) + B\sin(2t)) + 3\frac{d}{dt}(A\cos(2t) + B\sin(2t)) + 2(A\cos(2t) + B\sin(2t)) = 14\cos(2t) + 2\sin(2t) \\ \iff \quad & -4A\cos(2t) - 4B\sin(2t) + 3(-A\sin(2t) - 2B\cos(2t)) + 2(A\cos(2t) + B\sin(2t)) = 14\cos(2t) + 2\sin(2t) \\ \iff \quad & (-2A - 6B)\cos(2t) + (-6A - 2B)\sin(2t) = 14\cos(2t) + 2\sin(2t) \end{aligned}$$

Thus we conclude that we need

$$\begin{cases} -2A + 6B = 14 \\ -6A - 2B = 2 \end{cases}$$

Now we can solve this system in equations in our favorite way. For example, multiplying the second equation by 3 and adding the first yields

$$(-2A + 6B) + (-18A - 6B) = 14 + 2 \cdot 3 \implies -20A = 20 \implies A = -1.$$

Then we can use the second equation to conclude $2B = -6A - 2 = -6(-1) - 2 = 4$ and so $B = 2$. Thus $A = -1$ and $B = 2$ and therefore the particular solution is

$$y_p(t) = -\cos(2t) + 2\sin(2t).$$

To find the general solution, we must first find the general solution to the corresponding homogeneous equation

$$y'' + 3y' + 2y = 0.$$

The corresponding polynomial equation is

$$\lambda^2 + 3\lambda + 2 = 0,$$

which has solutions $\lambda = -1$ and $\lambda = -2$, so the general solution to the homogeneous differential equation is

$$y_h(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

Therefore the general solution to (81) is

$$y(t) = -\cos(2t) + 2\sin(2t) + C_1 e^{-t} + C_2 e^{-2t}.$$

Example 3. Find just one particular solution to

$$y'' + 25y = 10\cos(5t).$$

Solution. We could try our strategy of guessing $A\cos(5t) + B\sin(5t)$, but note that when we plug this in to the differential equation we get

$$\frac{d^2}{dt^2}(A\cos(5t)+B\sin(5t))+25(A\cos(5t)+B\sin(5t)) = -25(A\cos(5t)+B\sin(5t))+25(A\cos(5t)+B\sin(5t)) = 0.$$

Unfortunately, in this case our usual guess always yields 0 and therefore no choice of A and B will work. In this situation, we should guess something of the form

$$At\cos(5t) + Bt\sin(5t).$$

Indeed, compute

$$\begin{aligned}
& \frac{d^2}{dt^2}(At\cos(5t) + Bt\sin(5t)) + 25(At\cos(5t) + Bt\sin(5t)) = 10\cos(5t) \\
\iff \quad & \frac{d}{dt}((A + 5Bt)\cos(5t) + (B - 5At)\sin(5t) + 25(At\cos(5t) + Bt\sin(5t)) = 10\cos(5t) \\
\iff \quad & (10B - 25At)\cos(5t) + (-10A - 25Bt)\sin(5t) + 25(At\cos(5t) + Bt\sin(5t)) = 10\cos(5t) \\
\iff \quad & 10B\cos(5t) - 10A\sin(5t) = 10\cos(5t)
\end{aligned}$$

Thus we require $A = 0$ and $B = 1$ and so the particular solution is

$$y_p(t) = t\sin(5t).$$

Example 4. Find one particular solution to

$$y'' - 5y' + 6y = 4te^t.$$

Solution. In this case, we will guess the particular solution is of the form $(At + B)e^t$ in order to get $4te^t$ on the right side for some A and B. We plug in this expression to the differential equation to solve for which A and B work.

$$\begin{aligned}
& \frac{d^2}{dt^2}((At + B)e^t) + 5\frac{d}{dt}((At + B)e^t) + 6(At + B)e^t = 4te^t \\
\iff \quad & (At + B)e^t + 2Ae^t - 5(At + B)e^t - 5Ae^t + 6(At + B)e^t = 4te^t \\
\iff \quad & 2Ate^t + (2B - 3A)e^t = 4te^t
\end{aligned}$$

Thus this guess works exactly when

$$\begin{cases} 2B - 3A = 0 \\ 2A = 4 \end{cases}$$

which has solutions $A = 2$ and $B = 3$, so a particular solution is given by

$$y_p(t) = (2t + 3)e^t.$$

7/24 Mass-spring systems and damping

Springs. When you stretch a spring, it pulls back and the further you stretch a spring, the more it pulls. Similarly, when you compress a spring, it pushes back and the more you compress it, the harder it pushes. It turns out, the force the spring exerts is proportional to the distance you compress or stretch it from its equilibrium position (no stretching or compressing). This gives ***Hooke's Law***:

$$F = -kx \tag{82}$$

where x is the displacement of the spring from equilibrium, F is force of the spring trying to return to equilibrium, and the negative sign indicates that the direction of force will be in the opposite direction that you move the spring. The proportionality constant k is called the ***spring constant*** and depends how "stiff" the spring is: k will be very small for a slinky, but very large for a spring in the suspension system for a car. Recall by Newton's second law, force is mass times acceleration ($F = ma$) and so the units for force are $\text{kg}\cdot\text{m/s}^2$, which is abbreviated as one "Newton" and denoted by N. This implies the spring constant k has units of N/m.

Warm up. Suppose a spring is hanging from the ceiling. You attach a 1 kg weight to the spring, and the spring stretches 0.4 m. What is the spring constant k of the spring?

Solution. The force of gravity pulling down on the mass m is $F_g = mg$. The force of the spring up on the mass is $F_s = -kx$ where x is the displacement and k is the spring constant. These forces must balance. That is, we need

$$F_g + F_s = 0 \implies mg - kx = 0 \implies mg = kx.$$

We know $m = 1$ kg, $x = 0.4$ m, and $g \approx 10\,\text{m/s}^2$. Thus we get

$$1 \cdot 10 = k \cdot 0.4 \implies k = 25\,\text{N/m}.$$

Note that after hanging the mass on the spring, the spring still obeys Hooke's law – just with a new equilibrium position.

Simple Harmonic Motion. If a spring with spring constant k and with a mass m attached is displaced from equilibrium by some amount y, then Hooke's Law and Newton's second law give us the differential equation

$$m\ddot{y} = F_s = -ky.$$

That is,

$$m\ddot{y} + ky = 0 \tag{83}$$

If we are given the initial position and velocity of the spring, this differential equation allows us to solve for $y(t)$ for all time. The movement governed by a differential equation of this form is called ***simple harmonic motion***. While we will focus on the application of the mass-spring system, simple harmonic motion can also model pendulums, circuits, and many other phenomenon.

Example 1. Suppose our mass with $m = 1$ kg is hanging from our spring with $k = 25$ N/m. If we pull the spring a distance of 1 m and release, what is the motion $y(t)$ of the mass?

Solution. We must find the particular solution to the differential equation

$$\ddot{y} + 25y = 0.$$

with the initial conditions $y(0) = 1$ and $\dot{y}(0) = 0$. The characteristic polynomial equation is

$$\lambda^2 + 25 = 0,$$

which has solutions $\lambda = 5i$ and $\lambda = -5i$. Thus the general solution is

$$y(t) = C_1\cos(5t) + C_2\sin(5t).$$

The initial condition $y(0) = 1$ implies $C_1 = 1$. Next, compute

$$\dot{y}(t) = -5C_1\sin(5t) + 5C_2\cos(5t).$$

Then the initial condition $\dot{y}(0) = 0$ implies $0 = 5C_2 \implies C_2 = 0$. Thus the particular solution is

$$y(t) = \cos(5t).$$

Simple harmonic motion will be of the form $A\cos(\omega t - \phi)$ where A is the ***amplitude***, ω is the ***frequency***, and ϕ is the ***phase shift***. Furthermore, the value $T = 2\pi/\omega$ is the ***period*** of the motion which is how long it takes for the motion to make one cycle. In the example above, the amplitude is 1, the frequency is 5, the phase shift is 0, and the period is $2\pi/5$.

In general, if we have mass-spring system with mass m and spring constant k, then we study the differential equation

$$m\ddot{y} + ky = 0$$

which has characteristic polynomial equation

$$m\lambda^2 + k = 0$$

which has solutions $\lambda = \pm i\sqrt{\frac{k}{m}}$. If we let $\omega = \sqrt{\frac{k}{m}}$, then the general solution is

$$C_1\cos(\omega t) + C_2\sin(\omega t).$$

Observe that the frequency of this system is given by $\omega = \sqrt{\frac{k}{m}}$.

Damping. Next suppose a spring with spring constant k with a mass m is submerged in a liquid. This imposes a drag force proportional to the velocity of the mass $F_d = -\gamma\dot{y}$ where γ is the proportionality constant that depends on the liquid: for air, γ is very low and for honey γ is very high. Adding this to Newton's law gives

$$m\ddot{y} = F = F_d + F_s = -\gamma\dot{y} - ky$$

That is,

$$m\ddot{y} + \gamma\dot{y} + ky = 0. \tag{84}$$

The above differential equation can model damping in many different contexts. For example: there is a damping mechanism in a car's suspension system to reduce the oscillations for when a car rides over bumps, there is often a damping mechanism on doors to stop it from slamming shut, and there is an equivalent to damping in electrical circuits.

Once again, if we are given the initial displacement and velocity of this system we can predict the motion $y(t)$ for all time. However, the behavior of this motion can take a few different forms.

Example 2. Suppose $m = 1$, $k = 25$, and $\gamma = 8$. If we pull the spring a distance of 1 m and release, what is the motion $y(t)$ of the mass?

Solution. We must find the particular solution to the differential equation

$$\ddot{y} + 8\dot{y} + 25y = 0$$

with the initial conditions $y(0) = 1$ and $\dot{y}(0) = 0$. The characteristic polynomial equation is

$$\lambda^2 + 8\lambda + 25 = 0.$$

By the quadratic equation, this has solutions $-4 + 3i$ and $-4 - 3i$. Thus we have the general solution

$$y(t) = e^{-4t}(C_1\cos(3t) + C_2\sin(3t)).$$

The initial condition $y(0) = 1$ implies $C_1 = 1$. Next compute

$$\dot{y}(t) = -4e^{-4t}(C_1\cos(3t) + C_2\sin(3t)) + e^{-4t}(-3C_1\sin(3t) + 3C_2\cos(3t)).$$

So the initial condition $\dot{y}(0) = 0$ implies $0 = -4C_1 + 3C_2$. Thus $C_2 = 4/3C_1 = 4/3$. Therefore the particular solution is

$$y(t) = e^{-4t}(\cos(3t) + \frac{4}{3}\sin(3t)).$$

By the identity $A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t + \phi)$ for $A - Bi = re^{i\phi}$, we can rewrite this. First, write $1 - \frac{4}{3}i = \frac{5}{3}e^{i\arctan(-4/3)}$. Thus we have $\cos(3t) + \frac{4}{3}\sin(3t) = \frac{5}{3}\cos(3t + \arctan(-4/3))$ and so

$$y(t) = \frac{5}{3}e^{-4t}\left(\cos\left(3t + \arctan\frac{-4}{3}\right)\right).$$

Thus in this case we have sinusoidal motion that decays over time. If this modeled the damping mechanism on a door, the door would slam shut. The frequency of the cosine term of the decaying oscillations is called the ***quasifrequency*** of the motion (here the quasifrequency is 3).

Example 3. Suppose $m = 1$, $k = 25$, and $\gamma = 26$. If we pull the spring a distance of 1 m and release, what is the motion $y(t)$ of the mass?

Solution. We must find the particular solution to the differential equation

$$\ddot{y} + 26\dot{y} + 25y = 0.$$

with initial conditions $y(0) = 1$ and $\dot{y}(0) = 0$. The characteristic polynomial equation is

$$\lambda^2 + 26\lambda + 25 = 0$$

which has solutions $\lambda = -1$ and $\lambda = -25$. Thus the general solution is

$$y(t) = C_1e^{-t} + C_2e^{-25t}.$$

The initial condition $y(0) = 1$ implies $0 = C_1 + C_2$. Next, compute

$$\dot{y}(t) = -C_1e^{-t} - 25C_2e^{-25t}$$

and so the initial condition $\dot{y}(0) = 0$ implies $0 = -C_1 - 25C_2$. Adding these two equations together implies $1 = -24C_2$ and so $C_2 = -1/24$. Using $C_1 + C_2 = 0$ then implies $C_1 = 25/24$. Thus the particular solution is

$$y(t) = \frac{25}{24}e^{-t} - \frac{1}{24}e^{-25t}.$$

In this case, we have a slow exponential decay. If this modeled the damping mechanism on a door, the door would close very slowly.

Example 4. Suppose $m = 1$, $k = 25$, and $\gamma = 10$. If we pull the spring a distance of 1 m and release, what is the motion $y(t)$ of the mass?

Solution. We must find the particular solution to the differential equation

$$\ddot{y} + 10\dot{y} + 25y = 0$$

with initial conditions $y(0) = 1$ and $\dot{y}(0) = 0$. The characteristic polynomial equation is

$$\lambda^2 + 10\lambda + 25 = 0$$

which has only one solution: $\lambda = -5$. Thus the general solution is

$$y(t) = C_1e^{-5t} + C_2te^{-5t}.$$

The initial condition $y(0) = 1$ implies $1 = C_1$. Next compute

$$\dot{y}(t) = (-5C_1 + C_2)e^{-5t} + -5C_2te^{-5t}$$

and so the initial condition $\dot{y}(0) = 0$ implies $-5C_1 + C_2 = 0$ and so $C_2 = 5C_1 = 5$ which gives the particular solution

$$y(t) = e^{-5t} + 5te^{-5t}.$$

This is the choice of γ that leads to the fastest decay without sinusoidal motion. If this modeled the damping mechanism on a door, the door would close as quickly as possible without slamming.

Types of Damping. Given the general differential equation

$$m\ddot{y} + \gamma\dot{y} + ky = 0$$

the characteristic polynomial equation is

$$m\lambda^2 + \gamma\lambda + k = 0.$$

By the quadratic equation, this has solutions

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{-\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}.$$

We know from our studies of differential equations, that the behavior of the motion $y(t)$ depends greatly on these solutions λ. There are a few different cases

(1) If $(\gamma/(2m))^2 < k/m$ then we will have two complex solutions λ_1, λ_2 and so the motion $y(t)$ will have sinusoidal motion that decays over time. This is called ***under damping***.
(2) If $(\gamma/(2m))^2 > k/m$, then we have two real solutions λ_1, λ_2 and the solutions $y(t)$ will have slow exponential decay behavior. This is called ***over damping***.
(3) If $(\gamma/(2m))^2 = k/m$, then there is only one solution $\lambda_1 = \lambda_2$ and so the solution has relatively quick exponential decay behavior. This is called ***critical damping***.

7/26 Forcing, beats, and resonance

Forcing. We continue our study of mass-spring systems, but we will now additionally suppose there is some external force on the mass called the ***driving force***. We denote the additional force on $F(t)$ and in general the driving force will be a function of time. Adding this into Newton's laws we arrive at

$$m\ddot{y} = F = F_s + F_d + F(t) = -ky - \gamma\dot{y} + F(t).$$

Bringing the spring force and the drag force onto the left side of the equation we arrive at

$$m\ddot{y} + \gamma\dot{y} + ky = F(t). \tag{85}$$

Beats. We will suppose the driving force oscillates at a slightly different frequency than the natural frequency of the mass-spring system. This leads to interesting behavior called beats.

Example 1. Suppose we have a mass-spring system with $m = 1$ kg, $k = 25$ N/m, and no damping. Further, suppose we have a driving force of $F(t) = 11\cos(6t)$ Newtons. If the spring starts at rest, what will be the motion $y(t)$ of the spring?

Solution. We must find the solution to

$$\ddot{y} + 25y = 11\cos(6t). \tag{86}$$

such that $y(0) = 0$ and $\dot{y}(0) = 0$. Indeed, first we must find a particular solution. We guess that the particular solution is of the form $y_p(t) = A\cos(6t) + B\sin(6t)$. For this to be a solution, we need

$$\begin{aligned} \frac{d^2}{dt^2}(A\cos(6t) + B\sin(6t)) + 25(A\cos(6t) + B\sin(6t)) &= 11\cos(6t) \\ \iff \quad -11(A\cos(6t) + B\sin(6t)) &= 11\cos(6t) \end{aligned}$$

Thus we require $A = -1$ and $B = 0$. That is, $y_p(t) = -\cos(6t)$ is a particular solution. It only remains to find the general solution to the homogeneous differential equation

$$\ddot{y} + 25y = 0.$$

However, we have solved this last time! We know the general solution to the homogeneous equation is

$$y_h(t) = C_1\cos(5t) + C_2\sin(5t).$$

Therefore the general solution to (86) is

$$y(t) = -\cos(6t) + C_1\cos(5t) + C_2\sin(5t).$$

The initial condition $y(0) = 0$ implies $0 = -1 + C_1$ and so $C_1 = 1$. Next compute

$$\dot{y}(t) = 6\sin(6t) - 5C_1\sin(5t) + 5C_2\cos(5t)$$

and so $\dot{y}(0) = 0$ implies $5C_2 = 0$, so $C_2 = 0$. Thus the particular solution is

$$y(t) = \cos(5t) - \cos(6t).$$

We have found that the solution $y(t)$ is the difference of two cosine functions with a similar frequency. If we graph this, we see that there are periods of time where these two components align and make y very large. At other times, however, the two components are opposite and cancel. The result is sinusoidal motion whose amplitude is sinusoidal. We can use the identity $\cos(\theta) - \cos(\phi) = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$ to rewrite this as

$$y(t) = \cos(5t) - \cos(6t) = -2\sin\left(\frac{11}{2}t\right)\sin\left(\frac{-1}{2}t\right) = 2\sin\left(\frac{11}{2}t\right)\sin\left(\frac{1}{2}t\right).$$

Thus our solution is

$$y(t) = 2\sin\left(\frac{11}{2}t\right)\sin\left(\frac{1}{2}t\right).$$

Resonance. When the frequency of the forcing term perfectly aligns with the natural frequency of the oscillator, we find some interesting behavior of the solution $y(t)$.

Example 2. Suppose we have a mass-spring system with $m = 1$ kg, $k = 25$ N/m, and no damping. Further, suppose we have a driving force of $F(t) = 10\cos(5t)$ Newtons. If the spring starts at rest, what will be the motion $y(t)$ of the spring?

Solution. We must find the solution to the differential equation

$$\ddot{y} + 25y = 10\cos(5t) \tag{87}$$

with initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$. We must first find a particular solution to this differential equation. We have in fact studied this differential equation before and after guessing something of the form $At\cos(5t) + Bt\sin(5t)$, we found the particular solution to be

$$y_p(t) = t\sin(5t).$$

Furthermore, we have studied the corresponding homogeneous differential equation

$$\ddot{y} + 25y = 0$$

previously and found the general solution to this homogeneous differential equation to be

$$y_h(t) = C_1\cos(5t) + C_2\sin(5t).$$

Therefore the general solution to (87) is

$$y(t) = C_1\cos(5t) + C_2\sin(5t) + t\sin(5t).$$

The initial condition $y(0) = 0$ implies $C_1 = 0$. Compute the derivative

$$\dot{y}(t) = -5C_1\sin(5t) + 5C_2\cos(5t) + \sin(5t) - 5t\cos(5t),$$

and use the initial condition $\dot{y}(0) = 0$ to conclude $C_2 = 0$. Thus the motion $y(t)$ of the spring is

$$y(t) = t\sin(5t).$$

Plotting this, we find this gives rise to sinusoidal motion with amplitude that constantly increases.

8/02 Introduction to the Laplace Transform

The Laplace Transform is a powerful new way to solve differential equations. We motivate the formula with the following discussion about interest.

Interest. The amount of money $y(t)$ in a bank account with interest rate s (compounded continuously) is modeled by

$$\frac{dy}{dt} = sy.$$

Thus given some initial condition $y(0) = y_0$, solving for the amount $y(t)$ of money in the account amounts to solving an initial value problem

$$\begin{cases} \frac{dy}{dt} = sy \\ y(0) = y_0 \end{cases}$$

which we know has solution $y(t) = y_0 e^{st}$.

Problem. If you want to withdraw f dollars to be in your bank account after T years, how much should you deposit now?

Solution. We need to choose y_0 so that $f = y(T) = y_0 e^{sT}$ and so we need $y_0 = fe^{-sT}$.

Problem. Suppose you want to continuously withdraw from your bank account at a rate $f(t)$ that varies over time. What is the minimum you should deposit now to make this possible?

Solution. The amount $y(t)$ in the account is modeled by

$$\frac{dy}{dt} = sy - f(t).$$

We can rewrite this as

$$y' - sy = -f(t)$$

which we can solve using integrating factors. Indeed, multiply by $\mu(t)$ to get

$$\mu(t)y' - s\mu(t)y = -f(t)\mu(t).$$

We should choose the integrating factor $\mu(t)$ so that $\mu'(t) = -s\mu(t)$, which leads to the choice $\mu(t) = e^{-st}$. With this choice we can undo product rule and rewrite the differential equation as

$$\frac{d}{dt}(e^{-st}y) = -f(t)e^{-st}.$$

Now integrate both sides from 0 to ∞ and use $\lim_{t\to\infty} e^{-st}y(t) = 0$ to conclude that we should deposit

$$y(0) = \int_0^\infty f(t)e^{-st}dt.$$

That is, given the rate $f(t)$ of withdrawal, you should deposit $F(s) = \int_0^\infty f(t)e^{-st}dt$, which is a function of the interest rate.

The Laplace Transform. The ***Laplace transform*** of a function $f(t)$ is the function of s given by

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

The Laplace transform is often denoted using a capital letter as above or by

$$(\mathcal{L}f)(s) = \int_0^\infty f(t)e^{-st}dt.$$

Although we initially derived this formula by solving a finance problem, the true application of the Laplace transform is that is allows for a new technique to solve initial value problems. First, though, we will compute the Laplace Transforms of some functions.

Problem. Compute the Laplace transform of $f(t) = 1$.

Solution. We have

$$F(s) = (\mathcal{L}f)(s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}.$$

Problem. Compute the Laplace transform of $f(t) = t$.

Solution. Compute using integration by parts

$$F(s) = (\mathcal{L}f)(s) = \int_0^\infty te^{-st}dt = -\frac{1}{s}te^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = -\frac{1}{s^2}e^{-st}\Big|_0^\infty = \frac{1}{s^2}.$$

Problem. Compute the Laplace transform of $f(t) = t^2$.

Solution. Compute using integration by parts and using the above computation to get

$$F(s) = (\mathcal{L}f)(s) = \int_0^\infty t^2e^{-st}dt = -\frac{1}{s}t^2e^{-st} + \frac{2}{s}\int_0^\infty te^{-st}dt = \frac{2}{s^3}.$$

Problem. Compute the Laplace transform of $f(t) = e^{at}$ for some constant a.

Solution. Compute

$$F(s)(\mathcal{L}f)(s) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{(a-s)t}dt = \frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty = \frac{1}{s-a}$$

Problem. Compute the Laplace transform of $f(t) = \frac{1}{2}t^2 + 3e^t$.

Solution. Note we can break up this computation into Laplace transforms we already know:

$$\mathcal{L}\left(\frac{1}{2}t^2 + 3e^t\right) = \int_0^\infty \left(\frac{1}{2}t^2 + 3e^t\right)e^{-st}dt = \frac{1}{2}\int_0^\infty t^2e^{-st}dt + 3\int_0^\infty e^te^{-st}dt = \frac{1}{s^3} + \frac{3}{s-1}.$$

Linearity of Laplace Transform. The example above hints that the Laplace transform is *linear*. That is, the Laplace transform distributes over sums and scalar multiplication. To be precise, given any function $f(t)$ and $g(t)$ and any numbers a and b, then

$$\mathcal{L}(af + bg)(s) = \int_0^\infty (af(t) + bg(t))e^{-st}dt = a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt = a(\mathcal{L}f)(s) + b(\mathcal{L}g)(s).$$

That is, we have

$$\mathcal{L}(af + bg)(s) = a(\mathcal{L}f)(s) + b(\mathcal{L}g)(s). \tag{88}$$

Laplace Transform and Differentiation. The true power of the Laplace transform is in how it interacts with differentiation. For example, consider the following Laplace transforms that we have computed.

$f(t)$	$(\mathcal{L}f)(s)$
1	$1/s$
t	$1/s^2$
$t^2/2$	$1/s^3$

As we move up the left side of the table, we differentiate. However, as we move up the right side of the table, we do something much simpler: we multiply by s. Given any function $y(t)$, there is a simple way to compute $(\mathcal{L}y')(s)$ in terms of $(\mathcal{L}y)(s)$. Indeed, by integration by parts we get

$$(\mathcal{L}y')(s) = \int_0^\infty y'(t)e^{-st}dt = y(t)e^{-st}\big|_0^\infty + s\int_0^\infty y(t)e^{-st}dt = -y(0) + s\cdot(\mathcal{L}y)(s).$$

Therefore if we denote $Y(s) = (\mathcal{L}y)(s)$ we have the rule

$$(\mathcal{L}y')(s) = sY(s) - y(0). \tag{89}$$

We can also relate $(\mathcal{L}y'')(s)$ in terms of $(\mathcal{L}y)(s)$ by

$$(\mathcal{L}y'')(s) = s\cdot(\mathcal{L}y')(s) - y'(0) = s^2(\mathcal{L}y)(s) - sy(0) - y'(0).$$

Thus if we denote $Y(s) = (\mathcal{L}y)(s)$ we have

$$(\mathcal{L}y'')(s) = Y(s) - sy(0) - y'(0) \tag{90}$$

Transform Methods. Often times, transformations will turn harder problems into easy problems. For example, suppose you need to compute $\sqrt{7.6}$. One way to do this (and the way people did this before calculators) is to take the logarithm to get $\log(7.6)$ (by consulting a log table), then simply divide this value by 2 which gives $\frac{1}{2}\log(7.6) = \log(\sqrt{7.6})$. Then we just take the inverse logarithm of this result $\log(\sqrt{7.6})$ (again by consulting a log table) and we have computed $\sqrt{7.6}$.

We will see that because the Laplace transform takes differentiation to multiplication, the Laplace transform turns differential equations in algebra equations. That is, to compute the Laplace transform of the solution to a differential equation, we will just need to solve an algebra problem. Then we take the inverse Laplace transform to find the solution to the differential equation.

Applying Laplace transform to IVPs.

Example 1. Solve the IVP

$$\begin{cases} y'' - y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Solution. We can take the Laplace transform of both sides. Then using the notation $Y(s) = (\mathcal{L}y)(s)$ and the properties of the Laplace transform we get

$$\begin{aligned} & & y'' - y &= 0 \\ &\implies & \mathcal{L}(y'' - y)(s) &= 0 \\ &\implies & s^2Y(s) - sy(0) - y'(0) - Y(s) &= 0 \\ &\implies & (s^2 - 1)Y(s) &= s + 1 \\ &\implies & Y(s) &= \frac{1}{s-1}. \end{aligned}$$

Thus because $Y(s) = 1/(s-1)$, we conclude we must have $y(t) = e^t$.

Note on injectivity. In order to conclude that $Y(s) = 1/(s-1)$ implies $y(t) = e^t$ we used the fact that the Laplace transform is "injective", meaning two different functions cannot have the same Laplace transform. That is, if $(\mathcal{L}y_1)(s) = (\mathcal{L}y_2)(s)$, then $y_1(t) = y_2(t)$.

Example 2. Solve the IVP

$$\begin{cases} \ddot{y} + 26\dot{y} + 25y = 0 \\ y(0) = 1 \\ \dot{y}(0) = 0 \end{cases}$$

Solution. We take the Laplace transform of both sides of the differential equation which gives

$$\begin{aligned} & \ddot{y} + 26\dot{y} + 25y = 0 \\ \implies \quad & (s^2Y(s) - sy(0) - \dot{y}(0)) + 26(sY(s) - y(0)) + 25Y(s) = 0 \\ \implies \quad & (s^2 + 26s + 25)Y(s) = s + 26 \\ \implies \quad & Y(s) = \frac{s+26}{(s+1)(s+25)} \end{aligned}$$

Now we use partial fractions to rewrite $Y(s)$ in a simpler form. Indeed, we need to find the numbers A and B so that

$$\frac{A}{s+1} + \frac{B}{s+25} = \frac{s+26}{(s+1)(s+25)}.$$

This is the case exactly when $A(s+25) + B(s+1) = s + 26$ for all s. Plugging in $s = -1$ we see that $24A = 25$ and so $A = 25/24$. Plugging in $s = -25$ we see that $-24B = 1$ and so $B = -1/24$. Thus we conclude

$$Y(s) = \frac{25}{24} \cdot \frac{1}{s+1} + \frac{-1}{24} \cdot \frac{1}{s+25}.$$

The function that has this as its Laplace transform is

$$y(t) = \frac{25}{24} \cdot e^{-t} - \frac{1}{24}e^{-25t}.$$

Note that we have studies this initial value problem previously, and this process gives us the same answer!

Integral Transforms. In general, an ***integral transform*** takes as input a function $f(t)$ and outputs some function $(Tf)(s)$ by the rule

$$(Tf)(s) = \int_a^b f(t)K(t,s)dt$$

for some choice of ***integral kernel*** $K(t,s)$. In the case of the Laplace transform, we took $a = 0$, $b = \infty$, and $K(t,s) = e^{-ts}$. The Laplace transform has the extremely useful property that it takes differentiation to multiplication: if $f(0) = 0$, then $(\mathcal{L}f')(s) = s \cdot (\mathcal{L}f)(s)$. In fact, we will see the Laplace transform is the natural choice of transform of the form

$$(Tf)(s) = \int_0^\infty K(s,t)f(t)dt.$$

that has this property for all f with $f(0) = 0$. Indeed, if T is any transform as above so that $(Tf')(s) = s \cdot (Tf)(s)$, then we require

$$\int_0^\infty K(t,s)f'(t)dt = s \cdot \int_0^\infty K(t,s)f(t)dt.$$

We want this to hold for *all* f so that $f(0) = 0$ and so we consider f such that eventually $f(t) = 0$ for t large enough. Then we have

$$\int_0^\infty K(t,s)f'(t)dt = K(t,s)f(t)|_0^\infty - \int_0^\infty \frac{d}{dt}K(t,s)f(t)dt = -\int_0^\infty \frac{d}{dt}K(t,s)f(t)dt$$

Thus we need

$$-\int_0^\infty \frac{d}{dt}K(t,s)f(t)dt = s \cdot \int_0^\infty K(t,s)f(t)dt$$

for all f with $f(0) = 0$ and $f(t) = 0$ eventually for large t. The only hope we have for the above to be true for *all* such f is if

$$\frac{d}{dt}K(t,s) = -sK(t,s).$$

For each s, the function $K(t,s)$ is just a function of t, and we have seen this differential equation before: for each individual s, this has general solution

$$K(t,s) = Ce^{-ts}$$

The natural choice of $C = 1$ is precisely the kernel for the Laplace transform!

8/05 Table of Laplace transforms and applications to IVPs

We have computed some basic Laplace transforms such as

$$\mathcal{L}(1)(s) = \frac{1}{s}, \qquad \mathcal{L}(t)(s) = \frac{1}{s^2}, \qquad \mathcal{L}(t^2)(s) = \frac{2}{s^3}, \qquad \mathcal{L}(e^{at})(s) = \frac{1}{s-a}.$$

Next, we will compute the Laplace transforms of more functions, which will allow us to use the Laplace transform to solve more IVPs.

Laplace transform of $\mathbf{t^n}$. We have computed the Laplace transform $\mathcal{L}(1)(s) = 1/s$ as well as $\mathcal{L}(t)(s) = 1/s^2$ and $\mathcal{L}(t^2)(s) = 2/s^3$. In general, the Laplace transform of t^n is given by

$$\mathcal{L}(t^n)(s) = \int_0^\infty t^n e^{-st} dt = -\frac{1}{s} t^n e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt = \frac{n}{s}\mathcal{L}(t^{n-1})(s).$$

We have found a helpful relationship between $\mathcal{L}(t^n)(s) = \frac{n}{s}\mathcal{L}(t^{n-1})(s)$. For example, because $\mathcal{L}(t^0)(s) = 1/s$ this gives

$$\mathcal{L}(t)(s) = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}.$$

This also implies

$$\mathcal{L}(t^2)(s) = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}.$$

We can continue to get

$$\mathcal{L}(t^3)(s) = \frac{3}{s}\cdot\frac{2}{s^3} = \frac{3\cdot 2}{s^4}.$$

$$\mathcal{L}(t^4)(s) = \frac{4}{s}\cdot\frac{3\cdot 2}{s^4} = \frac{4\cdot 3\cdot 2}{s^5}.$$

Continuing this pattern, we find that in general we have

$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}.$$

Laplace transform of cosh(at) and sinh(at). Recall the functions $\cosh\theta$ and $\sinh\theta$ are given by

$$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2} \qquad \text{and} \qquad \sinh\theta = \frac{e^\theta - e^{-\theta}}{2}.$$

Because we know how to compute the Laplace transform of exponentials, we can also compute the Laplace transform of cosh and sinh. For example,

$$\mathcal{L}(\cosh(at))(s) = \mathcal{L}\left(\frac{e^{at}+e^{-at}}{2}\right)(s) = \frac{1}{2}\mathcal{L}(e^{at})(s) + \frac{1}{2}\mathcal{L}(e^{-at})(s) = \frac{1}{2}\cdot\frac{1}{s-a} + \frac{1}{2}\cdot\frac{1}{s+a} = \frac{s}{s^2-a^2}.$$

A similar computation gives that

$$\mathcal{L}(\sinh(at))(s) = \frac{a}{s^2-a^2}.$$

Laplace transform of cos(t) and sin(t). By using complex numbers to express cos and sin in terms of exponentials and that the Laplace transform of e^{at} we computed earlier works for complex values of a, we can compute the Laplace transforms of cos and sin in much the same way we computed the Laplace transforms of cosh and sinh.

$$\mathcal{L}(\cos(at))(s) = \mathcal{L}\left(\frac{e^{iat}+e^{-iat}}{2}\right)(s) = \frac{1}{2}\mathcal{L}(e^{iat})(s) - \frac{1}{2}\mathcal{L}(e^{-iat})(s) = \frac{1}{2}\cdot\frac{1}{s-iat} + \frac{1}{2}\cdot\frac{1}{s+iat} = \frac{s}{s^2+a^2}.$$

A similar computation gives that

$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}.$$

Laplace transform of $\mathbf{e^{at} \cdot f(t)}$. If we know the Laplace transform of any function $f(t)$, we can also compute the Laplace transform of $e^{at}f(t)$ by

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s-a)t}dt = \mathcal{L}(f)(s-a)$$

That is, if we denote $F(s) = (\mathcal{L}f)(s)$, then

$$\mathcal{L}(e^{at}f(t)) = F(s-a). \tag{91}$$

A case of this that appears frequently is when $f(t) = t^n$. Then we find

$$\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}. \tag{92}$$

Laplace transform of $\mathbf{t^n \cdot f(t)}$. Consider some function $f(t)$ and denote $F(s) = (\mathcal{L}f)(s)$. Then we compute how the derivative $F'(s)$ is related to $f(t)$ by

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty -tf(t)e^{-st}dt = -\mathcal{L}(tf(t)).$$

That is,

$$\mathcal{L}(tf(t)) = -F'(s).$$

From this, we can also compute

$$\mathcal{L}(t^2 f(t)) = -\frac{d}{ds}\mathcal{L}(tf(s))(s) = -\frac{d}{ds}(-1)\frac{d}{ds}F(s) = F''(s).$$

In general, by repeatedly taking derivatives and multiplying by -1 we find

$$\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s). \tag{93}$$

Table of Laplace Transforms. We summarize our computations in the following table. We let $f(t)$ be an arbitrary function and we denote $F(s) = (\mathcal{L}f)(s)$.

$y(t)$	$(\mathcal{L}y)(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$

Application to IVPs.

Example 1. Consider the initial value problem

$$\begin{cases} \ddot{y} + 25y = 11\cos(6t) \\ y(0) = 0 \\ \dot{y}(0) = 0. \end{cases}$$

Solution. We solve this using the Laplace transform, so first take the Laplace transform of both sides.

$$\begin{aligned} \mathcal{L}(\ddot{y} + 25y)(s) &= \mathcal{L}(11\cos(6t))(s) \\ \implies \quad (s^2Y(s) - sy(0) - y'(0)) + 25Y(s) &= \frac{11s}{s^2+36} \\ \implies \quad (s^2+25)Y(s) &= \frac{11s}{s^2+36} \\ \implies \quad Y(s) &= \frac{11s}{(s^2+36)(s^2+25)} \end{aligned}$$

Now we again apply partial fractions to simply $Y(s)$. Indeed, we are looking for constants so that

$$\frac{A_1s + B_1}{s^2+36} + \frac{A_2s+B_2}{s^2+25} = \frac{11s}{(s^2+36)(s^2+25)}.$$

By cross multiplying, we require

$$(A_1s + B_1)(s^2+25) + (A_2s + B_2)(s^2+36) = 11s.$$

Now we could distribute to get a system of 4 equations with 4 unknowns, but a shortcut is to plug in $s = 5i$ to the above equation and conclude

$$(A_2 5i + B_2)(11) = 11 \cdot 5i \implies A_2 = 1 \quad \text{and} \quad B_2 = 0.$$

Similarly, we can plug in $s = 6i$ to get

$$(A_1 6i + B_1)(-11) = 11 \cdot 6i \implies A_1 = -1 \quad \text{and} \quad B_1 = 0.$$

Thus we rewrite

$$Y(s) = \frac{-1}{s^2+36} + \frac{1}{s^2+25}.$$

The function that has this as its Laplace transform is

$$y(t) = \cos(5t) - \cos(6t).$$

Note we have solved this initial value problem before, and this method gives the same result.

Example 2. Solve the initial value problem

$$\begin{cases} y'' - 2y' + y = 2e^t \\ y(0) = 1 \\ y'(0) = -1. \end{cases}$$

Solution. We take the Laplace transform of both sides which gives

$$\begin{aligned} \mathcal{L}(y'' - 2y' + y)(s) &= \mathcal{L}(2e^t)(s) \\ \implies \quad (s^2Y(s) - sy(0) - y'(0)) - 2(sY(s) - y(0)) + Y(s) &= \frac{2}{s-1} \\ \implies \quad (s-1)^2Y(s) &= \frac{2}{s-1} + s - 3 \\ \implies \quad Y(s) &= \frac{2}{(s-1)^3} + \frac{s-3}{(s-1)^2}. \end{aligned}$$

Now we apply partial fractions to rewrite

$$\frac{s-3}{(s-1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2}.$$

By cross multiplying, we see that we need

$$s - 3 = (s-1)A + B.$$

That is, we need $s - 3 = sA + (B - A)$ and so A and B must satisfy the system

$$\begin{cases} A = 1 \\ A - B = 3. \end{cases}$$

Therefore we conclude $A = 1$ and $B = -2$. Therefore we can rewrite $Y(s)$ as

$$Y(s) = \frac{2}{(s-1)^3} + \frac{-2}{(s-1)^2} + \frac{1}{s-1}.$$

Using the formula for $\mathcal{L}(t^n e^{at})$, we see that the function that has this as its Laplace transform is

$$y(t) = t^2 e^t - 2te^t + e^t.$$

8/07 Piecewise forcing functions

We now study initial value problems of the non-homogeneous differential equations $ay'' + by' + cy = f(t)$ where the forcing function $f(t)$ is defined piecewise, which we can solve with the Laplace transform method!

Piecewise functions. A simple example of a piecewise function is

$$u_a(t) = \begin{cases} 0 & \text{if} \quad t < a \\ 1 & \text{if} \quad t \geq a. \end{cases}$$

This is called the ***Heaviside step function*** and all other piecewise functions can be built out of functions of this form. For example, consider the piecewise function

$$g(t) = \begin{cases} 1 & \text{if} \quad 1 \leq t < 2 \\ 0 & \text{otherwise.} \end{cases}$$

This can be written as $g(t) = u_1(t) - u_2(t)$. As another example, consider

$$h(t) = \begin{cases} t & \text{if} \quad t \geq 1 \\ 0 & \text{if} \quad t < 1. \end{cases}$$

Thus function can be written as $h(t) = tu_1(t)$. In general, for any function $f(t)$, the function $u_a(t)$ flattens out everything for $t < a$ and keeps all the values for $t \geq a$ the same. That is,

$$u_a(t)f(t) = \begin{cases} 0 & \text{if} \quad t < 1 \\ f(t) & \text{otherwise.} \end{cases}$$

We can make any piecewise function by adding together terms of this form $u_a(t)f(t)$.

Laplace transforms of piecewise functions. In order to solve $ay'' + by' + cy = f(t)$ for a piecewise function $f(t)$, we need to know how to take the Laplace transform of a piecewise function. The first step is to compute $\mathcal{L}(u_a(t))(s)$.

$$\mathcal{L}(u_a(t))(s) = \int_0^\infty u_a(t)e^{-st}dt = \int_a^\infty e^{-st}dt.$$

We can change the lower bound of this integral back to 0 by making the substitution $\tau = t - a$ so that $dt = d\tau$ and the integral becomes

$$\mathcal{L}(u_a(t))(s) = \int_0^\infty e^{-s(\tau+a)}d\tau = e^{-sa}\int_0^\infty e^{-s\tau}d\tau = e^{-sa}\mathcal{L}(1)(s) = e^{-sa} \cdot \frac{1}{s}.$$

That is,

$$\mathcal{L}(u_a(t))(s) = e^{-sa} \cdot \frac{1}{s}. \tag{94}$$

Because every piecewise function is built up of terms of the form $u_a(t)f(t)$, we need to be able to compute these Laplace transforms $\mathcal{L}(u_a(t)f(t))$. This can be done as follows.

$$\mathcal{L}(u_a(t)f(t)) = \int_0^\infty u_a(t)f(t)e^{-st}dt = \int_a^\infty f(t)e^{-st}dt.$$

Now again we make the substitution $\tau = t - a$ so that $dt = d\tau$ and the integral becomes

$$\mathcal{L}(u_a(t)f(t))(s) = \int_0^\infty f(\tau + a)e^{-s(\tau+a)}d\tau = e^{-sa}\int_0^\infty f(\tau + a)e^{-s\tau}d\tau = e^{-sa}\mathcal{L}(f(\tau + a))(s).$$

That is,

$$\mathcal{L}(u_a(t)f(t))(s) = e^{-sa}\mathcal{L}(f(t + a))(s). \tag{95}$$

If we denote $F(s) = (\mathcal{L}f)(s)$, then we can rewrite the above in the following nicer form

$$\mathcal{L}(u_a(t)f(t - a))(s) = e^{-sa}F(s). \tag{96}$$

With this rule, we can compute the Laplace transform of any piecewise function so long as we know the Laplace transforms of the functions we use to build the piecewise function. We should update our table of Laplace transforms with these additional rules.

$y(t)$	$(\mathcal{L}y)(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$u_a(t)$	e^{-sa}/s
$u_a(t)f(t-a)$	$e^{-sa}F(s).$

Inverse Laplace transforms. We can use (94) and (95) to compute the inverse Laplace transforms of additional functions.

Example. Find the function with Laplace transform $e^{-sa}/(s-b)$.

Solution. We know $f(t) = e^{bt}$ has Laplace transform $F(s) = 1/(s-b)$ and so by (96) we know

$$\mathcal{L}\left(u_a(t)e^{b(t-a)}\right)(s) = e^{-sa}/(s-b).$$

Example. Find the function with Laplace transform $e^{-sa} \cdot s/(s^2+b^2)$.

Solution. Once again we know $f(t) = \cos(bt)$ has Laplace transform $F(s) = s/(s^2+b^2)$, and so again by (96) we know

$$\mathcal{L}\left(u_a(t) \cdot \cos(b(t-a))\right)(s) = e^{-sa} \cdot \frac{s}{s^2+b^2}$$

Example. Find the function with Laplace transform $e^{-sa} \cdot b/(s^2+b^2)$.

Solution. We know $f(t) = \sin(bt)$ has Laplace transform $F(s) = b/(s^2+b^2)$ and so by (96) we know

$$\mathcal{L}(u_a(t)\sin(b(t-a)))(s) = e^{-sa} \cdot \frac{b}{s^2+b^2}$$

Application to IVPs.

Example. Solve the following IVP

$$\begin{cases} y'' + y = f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

where $f(t)$ is the square wave pulse function

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \leq t < \pi, \\ -1 & \text{if} \quad \pi \leq t < 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

We can rewrite this using Heaviside step functions as $f(t) = u_0(t) - 2u_\pi(t) + u_{2\pi}(t)$. Thus we must solve the differential equation

$$y'' + y = u_0(t) - 2u_\pi(t) + u_{2\pi}(t)$$

Taking the Laplace transform of both sides and solving for the Laplace transform $Y(s) = (\mathcal{L}y)(s)$ gives

$$\begin{aligned} & \mathcal{L}(y'' + y)(s) = \mathcal{L}(u_0(t) - 2u_\pi(t) + u_{2\pi}(t))(s) \\ \Longrightarrow \quad & (s^2+1)Y(s) = \frac{1}{s} - \frac{2}{s}e^{-\pi s} + \frac{1}{s}e^{-2\pi s} \\ \Longrightarrow \quad & Y(s) = \frac{1}{s(s^2+1)}(1 - 2e^{-\pi s} + e^{-2\pi s}). \end{aligned}$$

Next we apply partial fractions to split the above into terms that we can compute the inverse Laplace transform of. By partial fractions there exists constants A, B, and C so that

$$\frac{1}{s(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s}.$$

Cross multiplying reveals we require

$$1 = s(As + B) + (s^2+1)C$$

Substituting $s = 0$ implies $C = 1$. Then we need $A = -1$ and $B = 0$ which yields

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2+1)}(1 - 2e^{-\pi s} + e^{-2\pi s}) \\ &= \left(\frac{1}{s} - \frac{s}{s^2+1}\right)(1 - 2e^{-\pi s} + e^{-2\pi s}) \\ &= \frac{1}{s} - \frac{2}{s}e^{-\pi s} + \frac{1}{s}e^{-2\pi s} - \frac{s}{s^2+1} + 2e^{-\pi s}\frac{s}{s^2+1} - e^{-2\pi s}\frac{s}{s^2+1}. \end{aligned}$$

Thee function $y(t)$ that has the above expression as its Laplace transform is

$$\begin{aligned} y(t) &= 1 - 2u_\pi(t) + u_{2\pi}(t) - \cos(t) + 2u_\pi(t)\cos(t-\pi) - u_{2\pi}(t)\cos(t - 2\pi) \\ &= 1 - 2u_\pi(t) + u_{2\pi}(t) - \cos(t) - 2u_\pi(t)\cos(t) - u_{2\pi}(t)\cos(t). \end{aligned}$$

8/09 Dirac delta forcing functions

Dirac delta function. Consider the velocity and acceleration plotted in Figure 10.

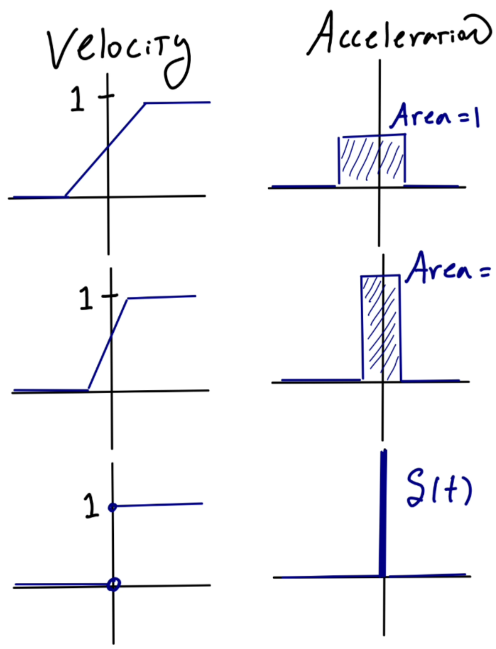


Figure 10. Velocity and acceleration

If the velocity $v(t)$ increases to 1, the area under the acceleration curve $a(t) = v'(t)$ is 1. If the velocity increases to 1 over a shorter time interval, then the acceleration curve is narrower and taller, still with area 1 under the curve. If the velocity $v(t)$ changes *instantaneously*, then we call the resulting acceleration the "dirac-delta distribution" which is denoted $\delta(t)$. That is, the dirac-delta is the derivative of the Heaviside function $\delta(t) = \frac{d}{dt}u_0(t)$, and $\delta(t) = 0$ for all $t \neq 0$ but still $\int \delta(t)dt = 1$. This mathematical object is not a function in the usual sense, but it is still a useful concept to capture instantaneous acceleration. For another application, consider Figure 11, which plots the total amount of money withdrawn from a bank account, and the rate of withdrawal.

Once again, observe the faster you widthraw money from the account, the taller and narrower the rate of withdrawal plot is. If you withdraw \$1 in total, then the area under the rate of withdrawal curve is 1. If you withdraw \$1 all at once at time c, then the rate of withdrawal is again given by the dirac-delta, but now shifted to be centered at c, which we denote as $\delta_c(t) = \delta(t - c)$. Note we have $\delta_c = \frac{d}{dt}u_c(t)$.

Intuitively, we can think of $\delta_c(t)$ as an arbitrarily tall and narrow function with area 1. Note therefore that we find

$$\int_{-\infty}^{\infty} \delta_c(t)f(t)dt = f(c) \tag{97}$$

because multiplying a tall and skinny function with area 1 by centered at c by $f(t)$ simply scales this function by $f(c)$ vertically and this $\delta_c(t)f(t)$ will have area $f(c)$. In fact, it is helpful to define the ***dirac-delta distribution*** as the object that makes (97) true for all functions $f(t)$.

Laplace transform of delta function. Next we ask, what is the Laplace transform of $\delta_c(t)$? Recall the Laplace transform of $f(t)$ at s represents the minimum amount of money at $t = 0$ we must have in our bank account to withdraw money at a rate $f(t)$ with interest rate s. Thus the Laplace transform of $\delta_c(t)$ at s represents how much money we need to have in our bank account if we want to withdraw \$1 all at once at time t. In fact, we solved this when we introduced the Laplace transform: we need $\$e^{-cs}$ in our bank account

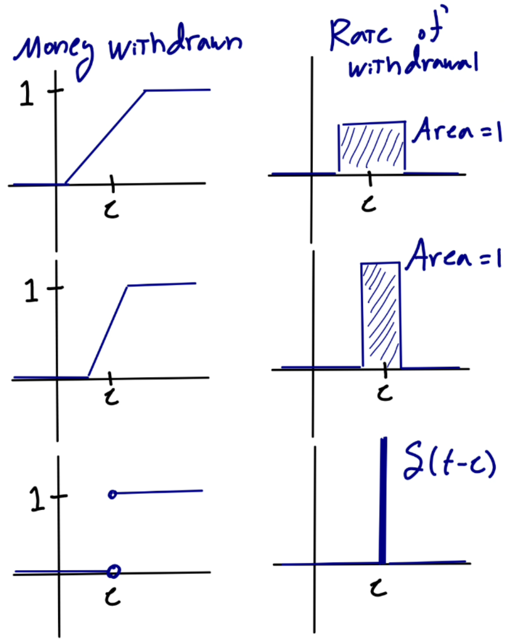


FIGURE 11. Withdrawing money from an account

initially. Thus we expect $\mathcal{L}(\delta_c(t))(s) = e^{-cs}$. Indeed, note we can use the definition (97) of dirac-delta to compute

$$\mathcal{L}(\delta_c(t))(s) = \int_0^\infty \delta_c(t)e^{-st}dt = \int_{-\infty}^\infty \delta_c(t)u_0(t)e^{-st}dt = u_0(c)e^{-sc} = e^{-cs}.$$

This formally confirms our suspicion, that

$$\mathcal{L}(\delta_c(t))(s) = e^{-cs}. \tag{98}$$

Application to IVPs. The dirac-delta can represent a forcing function in an initial value problem when we want to model near-instantaneous acceleration. For example, consider a mass with $m = 1$ kg on a spring with spring constant $k = 1$ N/m at rest. If at $t = 2$ s we suddenly hit this spring with a baseball bat, we might model this as the initial value problem

$$\begin{cases} y'' + y = \delta_2(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Because we can take the Laplace transform of $\delta_2(t)$, we can solve this IVP with the Laplace transform method. Indeed, compute the Laplace transform of both sides to get

$$\begin{aligned} & (s^2+1)Y(s) = e^{-2s} \\ \implies \quad & Y(s) = \frac{e^{-2s}}{s^2+1} \\ \implies \quad & y(t) = u_2(t)\sin(t-2). \end{aligned}$$

Thus the solution to this IVP is $y(t) = u_2(t)\sin(t-2)$. This is a function that is 0 for all time before 2, then oscillates after 2, which is the behavior we would expect from hitting a stationary mass on a spring with a baseball bat.

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