MATH 207A FINAL

INTRO TO DIFFERENTIAL EQUATIONS

AUGUST 16, 2024

NAME: ______________________________

Exam Guidelines.

- Write your name above.
- Remain on this title sheet until you are instructed to begin the exam.
- The time given for this exam is 60 minutes.
- There are 5 questions and each question is worth 10 points.
- You may not use any electronic device except for a non-graphing, non-programmable calculator (such as a TI30 or equivalent). A calculator is not necessary for this exam.
- For full credit, show your work for each problem. You may use the space directly below the problem statement and the blank page following each problem.
- Clearly mark your final answer. For questions with multiple parts, clearly mark which part you are working on.
- Let me know if you have a question about what a problem is asking. If you are taking this exam at a proctoring center, the proctor can direct questions to seanhr@uw.edu.

Good luck!

Laplace Transforms. We use the notation $F(s) = (\mathcal{L}f)(s)$ and $Y(s) = (\mathcal{L}y)(s)$ below.

$g(t)$	$(\mathcal{L}g)(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)$	$\frac{a}{s^2+a^2}$

$g(t)$	$(\mathcal{L}g)(s)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$u_a(t)$	e^{-sa}/s
$u_a(t)f(t-a)$	$e^{-sa}F(s)$
$u_a(t)f(t)$	$e^{-sa}\mathcal{L}(f(t+a))(s)$
$\delta(t-c)$	e^{-cs}
$y'(s)$	$sY(s) - y(0)$
$y''(s)$	$s^2Y(s) - sy(0) - y'(0)$

Trigonometric Identities.

$$\begin{aligned}
\cos(\theta) - \cos(\phi) &= -2\sin\left(\frac{\theta-\phi}{2}\right)\sin\left(\frac{\theta+\phi}{2}\right)\\
\sin(\theta) - \sin(\phi) &= 2\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)\\
\sin(\theta) + \sin(\phi) &= 2\cos\left(\frac{\theta-\phi}{2}\right)\sin\left(\frac{\theta+\phi}{2}\right)\\
\cos(\theta) + \cos(\phi) &= 2\cos\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)\\
A\cos(\theta) + B\sin(\theta) &= r\cos(\theta-\phi) \qquad \text{for} \quad A + Bi = re^{i\phi}.
\end{aligned}$$

Strategies to solve differential equations.

Separation of variables: If you can rewrite the differential equation as $g(y)y'(t) = h(t)$ (i.e. if the differential equation is separable), then integrate both sides $\int g(y)dy = \int h(t)dt$ and solve for y.

Integrating factors: If you can rewrite the differential equation as $y' + p(t)y = g(t)$ (i.e. if the differential equation is linear), multiply the equation by $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$. By undoing product rule, this can be rewritten $\frac{d}{dt}(\mu(t)y) = g(t)\mu(t)$. Now integrate both sides and solve for y.

2nd order constant coefficient: To solve $y'' + by' + cy = 0$, study solutions to $\lambda^2 + b\lambda + c = 0$.

2nd order non-homogeneous constant coefficient: The general solution to the non-homogeneous differential equation $y'' + by' + cy = f(t)$ is $y(t) = y_p(t) + y_h(t)$ where $y_p(t)$ is any particular solution to the differential equation, and $y_h(t)$ is the general solution to $y'' + by' + cy = 0$.

Laplace transform: To solve $ay'' + by' + cy = f(t)$ with $y(0) = y_0$, $y'(0) = v_0$, take the Laplace transform of both sides, solve for the Laplace transform $(\mathcal{L}y)(s)$ of the solution, then use this to find the solution $y(t)$.

Modeling with differential equations.

- Mixing problems. Mass $m(t)$ of salt satisfies $\frac{dm}{dt} = \text{rate salt comes in} - \text{rate salt goes out}$.
- Newton's law of cooling: temperature $T(t)$ of an object satisfies $\frac{dT}{dt} = -k(T - T_{\text{amb}})$.
- Radioactive decay: amount $A(t)$ of radioactive substance satisfies $\frac{dA}{dt} = -kA(t)$.
- Population dynamics: population $P(t)$ satisfies $\frac{dP}{dt} = rP$ by the simple growth model, and $\frac{dP}{dt} = rP(1 - P/M)$ by logistic growth model. Subtract rate of harvesting or add rate of adding population.
- Drag and terminal velocity: velocity $v(t)$ of falling object satisfies $\frac{dv}{dt} = g - \alpha v$.
- Mass-spring systems: m mass, γ drag coefficient, k spring constant, then $m\ddot{y} + \gamma\dot{y} + ky = F(t)$.

Problem 1 [10 points]. Solve the following IVP using the Laplace transform method.

$$\begin{cases} y'' + 4y = 4 \\ y(0) = 1 \\ y'(0) = 2. \end{cases}$$

Problem 2 [10 points]. Solve the following initial value problem.

$$\begin{cases} y'' + 3y' + 2y = \delta_1(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note: $\delta_1(t) = \delta(t-1)$ is the dirac-delta function centered at 1.

Problem 3 [10 points total]. Find the general solution to each of the following.

(a) [5 points] $y' = e^{t-y}$.

(b) [5 points] $y'' + 4y' + 5y = 20e^{t}$.

Problem 4 [10 points]. You start with some volume V_0 of fresh water in a tank with a capacity of 4 L. Next, salt water with concentration 1 g/L is pumped into the tank at a rate of 2 L/min and water is pumped out of the tank at a rate of 1 L/min. What volume V_0 of fresh water should you start with to have 3 g of salt in the tank when it reaches capacity?

Problem 5 [10 points total]. Solve the following initial value problem with two different methods. You can solve this problem with integrating factors (by first solving for y'), by finding the homogeneous and particular parts of the solution, and with the Laplace transform.

$$\begin{cases} y'' + y' = e^{-t} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

(a) [5 points] Solve the above with one of the three methods listed.

(b) [5 points] Solve the above with a different method.